Optimizing the Efficiency of a Multi-Stage Axial-Flow Compressor: An Application of Stage-Wise Optimization

THESIS

Shawn A. Miller, B.S.
2nd Lt, USAF

AFIT/GOR/ENS/98M-17

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY
AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

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Shawn A. Miller, B.S.

2nd Lt, USAF

Air Force Institute of Technology

Wright-Patterson AFB, Ohio

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Fan and Compressor Branch, Turbine Engine Division, Propulsion Directorate, Air Force Research Laboratory, Wright-Patterson AFB, OH (AFRL/POTF)

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Approved:

JAMES W. CHRISSIS
Associate Professor of Operations Research
Department of Operational Sciences
Advisor

12 Mar 98
Date

PAUL I. KING
Associate Professor of Aerospace Engineering
Department of Aeronautical and Astronautical Engineering
Advisor

16 Mar 98
Date

STUART C. KRAMER, Lt Col, USAF
Associate Professor of Aerospace Systems Engineering
Department of Aeronautical and Astronautical Engineering
Reader

12 MAR 98
Date

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Shawn A. Miller

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Nomenclature

Symbol	Definition
A	area (meters squared)
C	absolute velocity(meters per second)
c	chord (meters)
D_f	diffusion factor
h	blade height (meters)
i,δ	incidence and deflection angles (degrees)
M	Mach number
m	mass flow rate (kilograms per second)
P	pressure (kilograms per cubic meter)
r	radius (meters)
T	temperature (Kelvin)
U	rotational velocity (revolutions per second)
V	relative velocity (meters per second)
α, β	absolute and relative air angles (degrees)
λ	work done factor
Λ	degree of reaction
η	efficiency
ω	losses

Subscripts

Symbol	Definition
a	axial
act	actual
h	hub
m	mean
r	rotor
s	stator
st	stage
shk	shock
T	total
t	tip
th	theoretical
w	whirl
1,2,3.4.5	stations of stages

Superscript

Symbol	Definition
1,2	stages

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Abstract

The development of jet engines has become an integral part of maintaining air superiority. In order to achieve the most advanced engine, research has turned to traditional optimization methods to aid in creating new engine designs. To develop simplified mathematical models representative of the engine, the engine can be separated into its components. A jet engine has three major elements, the compressor, combustion chamber and turbine. This research attempts make an initial analysis of a two stage compressor to determine values of air angles and spacing to chord ratios for both stages that produce the highest possible efficiency for the overall two stage compressor. A pitchline model is developed representing the two stage compressor and is used in conjunction with a optimization method to solve for the on-design air angles and spacing to chord ratios. The results of the model were compared to examples available in current literature to ensure the model properly represented a compressor stage. The off -design performance of the results were calculated to determine how the on-design optimal designs would operate under off-design conditions. Since operational compressors are made up of many stages, analysis is performed to determine which optimization method would be most useful in determining a multistage compressor design.

Optimizing the Efficiency of a Multi-Stage Axial-Flow Compressor: An Application of Stage-Wise Optimization

Chapter 1 - Introduction

1.1 Background

Since the Wright brothers first flew in 1903, man has tried to fly higher and faster. From the 1930s to the 1950s, the introduction and subsequent wide application of the jet engine greatly increased the altitude and speed at which aircraft flew. From that time until today, the jet engine has become a main feature on aircraft. Early engine designs were based on experience of the designers. The designers would create designs that were more focused on being operational than optimal. It was a long process that wasn't focused on achieving an optimal design. Even today, engine design is mostly an iterative approach where the design is reevaluated and revised many times to establish a design that achieves a certain operating level with the design still being based on the designers' experience. A fundamental problem with basing the design solely on the designers' experience is that the design may not necessarily be "optimal". The question the designer should ask is: What is an optimal design given the engine's specific operating conditions or performance profile?

This question can be addressed by incorporating optimization techniques into the engine design process. For the jet engine as a whole, it would be difficult to develop a complete mathematical model for use in finding an optimal design due to the complexity of the engine. Currently, there is no model that represents all the aerodynamic, thermodynamic and structural interactions occurring in an engine. Since developing a model of the entire engine is extremely complex and not possible at this time, it might be simpler to develop models for various engine components and determine how the design of each component affects the performance of the engine. This "decomposing" of the engine into it's components allows for smaller, more manageable models representing each component to be developed. Once a model for each component is developed, an optimization approach can be used to determine an optimal design for each component. Analysis can then be performed to determine how the components' designs affect the engine's performance.

A jet engine can be represented as three separate components: the compressor, combustion chamber, and the turbine. The compressor and the turbine have many design variables associated with them that make it difficult, if not impossible, to enumerate all possible values of the design variables, even in a given feasible region. Formulating a mathematical model of the compressor or turbine and incorporating an optimization technique will help the designer develop a more efficient component. By optimizing each component, analysis could be performed to determine what effect the optimal components had on the overall design of the engine. There is a need to develop models for each component and to determine what, if any, optimization methods could be used to solve for the optimal design.

1.2 Research Purpose

The research described herein examines using an optimization approach to determine the optimal design for an axial-flow compressor. Some research (Massardo and Satta, 1990; Beknev, *et al.*, 1991) has already examined certain aspects of solving for the optimal design variables for a compressor. This research is an extension of previous research (Massardo and Satta, 1990) where only one stage was addressed in the model. By modeling a single stage, it ignores the importance of the stage's outlet conditions. In reality, the outlet conditions from one stage control the inlet conditions for the next stage. Therefore, this research models a two-stage axial-flow compressor. By looking at two stages, an understanding of how the previous stage affects the next stage is gained. Modeling two stages is also of value to determine how the efficiencies of each stage relate to the overall efficiency of the compressor. This research is to be the first step in incorporating an efficient optimization technique that can be used to determine an optimal design of an axial-flow compressor consisting of more than two stages. This is a necessity since most compressors used in aircraft today are made up of multiple stages.

1.3 Compressor Operating Theory

1.3.1 Fundamental Theory

There are two main types of compressors used in aircraft engines: centrifugal, and axial. Currently, axial-flow compressors are used exclusively in the design of high-performance jet engines. This is because axial-flow compressors can obtain a higher pressure ratio and greater efficiency than centrifugal compressors. Also, they provide a greater flow rate for a given frontal area (Cohen *et al.*, 1985). This research only examines axial-flow compressors since they are the main type used in aircraft today. For this reason, whenever discussing the axial-flow compressor, it will be simply referred to as just the *compressor*.

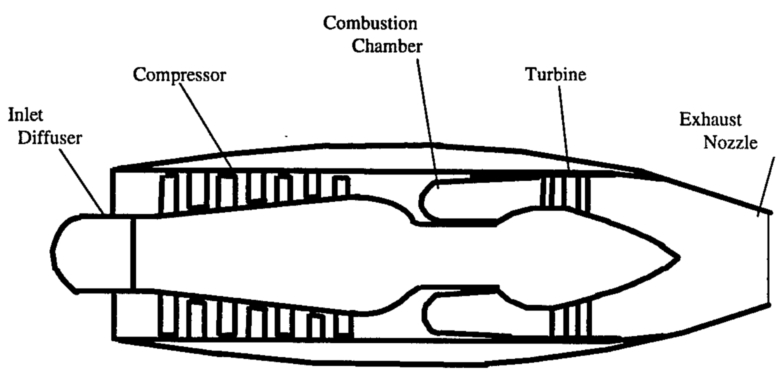

Figure 1. View of turbofan engine.

The location of the components of an engine are shown in Figure 1. The compressor is located in the forward area of a jet engine. A compressor does exactly what its name implies; it compresses the working fluid moving through the engine. The fluid in the case of a jet engine is air. By compressing the air, the pressure of the air is increased, allowing the air to be expanded in the turbine. If the air was sent directly to the turbine from the compressor, and there were no energy losses in either the compressor or the turbine, the power developed by the turbine would be equal to the power absorbed by the compressor. In order to achieve a positive energy output, a combustion chamber is added between the compressor and the turbine to raise the temperature of the air before it enters the turbine. Expansion of the heated air in the turbine yields a power output great enough to drive the compressor as well as provide thrust.

The compressor consists of multiple stages connected in series. Each stage contains two rows of blades known as the *rotor* and *stator*. The rotor is attached to a *driveshaft* that is driven by the turbine, while the stator is fixed to the outer wall of the compressor. A diagram of a single stage is shown in Figure 2.

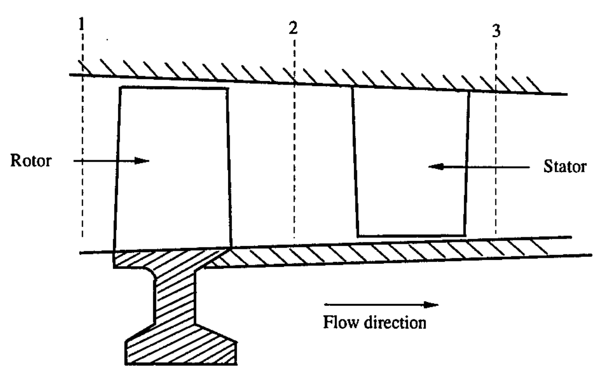

Figure 2. Stations of compressor stage.

Air enters a stage where it is accelerated by the rotor. The rotor increases the air's velocity as it travels to the stator. The increase in velocity also corresponds to an increase in the air's kinetic energy as it enters the stator. This kinetic energy is converted into static pressure as the air decreases velocity in the stator. The transfer of energy in the stage creates an increase in pressure. The stator sets up the distribution of airflow to enter the next stage at the proper air angle. This process is completed for every stage until the appropriate pressure rise is achieved.

The pressure increase across the stage creates an adverse pressure gradient in the compressor stage since the fluid flow is traveling in the direction of higher pressure. Since the absolute air velocity is increased as it enters the rotor, the relative velocity of the air to the rotor decreases. In other words, there is diffusion across the rotor in the stage. There is a limit on the diffusion that can occur in each stage, due to the small change of the annular flow area of the diffusing flow. The limitation of diffusion means there is little rise in the static pressure over the stage. This is why many stages are required to yield the needed pressure increase to operate a jet engine. The annulus flow area has to decrease in each successive stage. The reason for this is it is desirable to keep the axial-flow velocity of the air somewhat constant though the compressor. As the air gets compressed in each successive stage, the density of the air increases. To keep the air velocity approximately constant with an increasing density, the annulus flow area (and hence the blade diameters) have to be decreased. For information about compressor theory, refer to relevant literature such as Cohen (Cohen, *et al.*,1985), Horlock, (Horlock, 1973) or Lieblin (Lieblin, 1965)

These are the fundamentals of how a compressor operates to increase the pressure in the engine. Some special considerations need to be taken into account to properly model the compressor. They are presented in the next section.

1.3.2 Factors Affecting Compressor Operation

Two other considerations that need to be taken into account when modeling a compressor are: stalling of compressor stages, and high entry velocity flow in the compressor. These are serious complications that need to be properly taken into account when designing a compressor. Stalling of a compressor stage occurs when the difference between the flow direction and the blade angle, known as the *incidence angle*, becomes too large. This difference occurs since the rotor and stator of the stage are designed to operate for a specific set of operating conditions. In reality, the compressor encounters a wide range of operating conditions. These varying operating conditions change the velocity of fluid flow and produce a difference between the flow direction and the blade angle. It is important to design a compressor that operates at an acceptable efficiency over a wide range of operating conditions.

For air entering the compressor with a high velocity (on the order of Mach 0.8 or higher) stalling becomes an important modeling consideration. Two important effects take place as the Mach number increases: the overall losses in the compressor increase substantially, and the range of the incidence angle for which losses are acceptable is greatly reduced. The losses have been determined to be from shock waves as well as shock-boundary layer interactions that form in the blade row (Kinnebrock, 1981). At higher Mach numbers, the blade spacing also becomes an important modeling consideration to minimize the losses. It was shown in cascade testing that an increase in the pitch-to-chord ratio causes a rapid increase in the loss across the stage (Cohen *et al.*, 1985).

To account for stalling, the *off-design* of the compressor for a fixed geometry in a stage needs to be examined to determine the range or operating conditions where the stage maintains an acceptable efficiency. Compressor stages operating under transonic conditions have shock losses resulting from shock waves inside the compressor. To properly model the compressor, these considerations need to be taken into account. These details are shown in the development of the model in Chapter 2.

1.4 Problem Statement

The focus of this research is threefold: first, a mathematical model for the two-stage compressor is developed. Also, appropriate optimization methods for determining the optimal design for the two-stage compressor are addressed. Then, an attempt to make conclusions about the possibility of using the optimization technique for a multiple-stage compressor is discussed. To accomplish this, a mathematical model representing the design of the compressor is developed. The mathematical model is used to determine the decision variables that yield the highest value of the compressor efficiency. Since the model contains aerodynamic equations linking the state conditions at different points in the stages which are nonlinear equations, the optimization approach draws from the branch of optimization known as *nonlinear programming*. Therefore, the problem statement of this research is to develop a mathematical model of a two stage compressor and use the model to solve for the maximum overall efficiency of the two stages together using nonlinear programming. The objective function and model are used to determine which optimization method could be used to solve for a two stage compressor. Considerations are also made on the utility of the optimization technique to solve for the optimal design of a more general N-stage compressor.

1.5 Thesis Overview

The remainder of this thesis develops theory of the compressor model used as well as the optimization approaches taken into consideration to determine an optimal compressor design. Chapter Two introduces compressor design theory and how it was used to develop the compressor model used in this thesis. Chapter Three presents the possible optimization approaches considered with emphasis given to the approaches used in this thesis. The Forth chapter focuses on the results the nonlinear program yields for the on-design of the compressor as well as off-design implications of the design. The final chapter discusses the conclusions of this research and possibilities for future research in this area.

Chapter 2 - Compressor Design and Modeling

2.1 Introduction

This chapter introduces the necessary theory to develop a mathematical model of the two-stage compressor as well as presenting the model developed for use in this research. The introduction of the necessary theory provides the reader a basic understanding relating to important issues that need to be considered when developing a compressor model.

For the model used in this research, this chapter discusses the assumptions that were made, including reasons why these assumptions were made. Development of the constraints and the objective function for the two-stage model are also presented. At the end of the chapter, there is a brief discussion concerning validation of the model compared to available examples found in the literature.

2.2 Compressor Design Theory

2.2.1 On-Design

To exactly model a compressor stage would require accounting for the complicated three-dimensional flow of the air through the rotor and stator. *Time* would also have to be modeled since steady-state conditions could not be met due to the continuous changes of the air flow in the compressor. No model can be developed that takes into account all of these factors due to complexity of the air flow. Thus a simplified model had to be developed; approximations were made to allow for a partially-accurate model of the compressor in two dimensions. Since the pressure increase in each stage is small, the assumption of *incompressible flow* can be used to determine the changes of the fluid.

In practice the aerodynamic design of a multistage compressor may be considered to consist of three phases: 1) Determination of stage velocity diagrams for on-design, 2) Selection of stage blading, and 3) Determination of off-design performance (Lieblin, 1965).

Stage velocity diagrams enable the designer to determine the change in velocity of the air entering and leaving the rotor and stator of each stage. An example of velocity diagrams for a single stage are given in Figure 3.

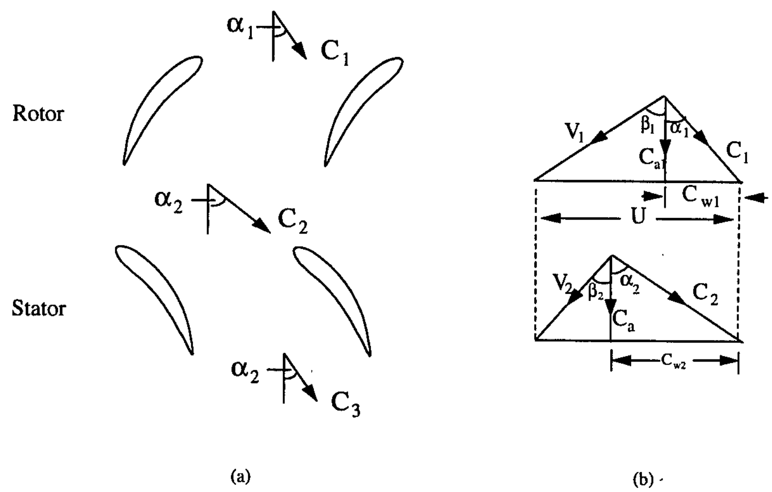

Figure 3. a) Absolute and relative air angles. b) Velocity diagrams.

In Figure 3, the αs and the Cs represent the absolute air angle and velocity, respectively, and the βs and the Vs represent the air angles and velocities relative to the rotor. When designing a compressor the inlet operating conditions are assumed to be known. This means α_1 and C_1 are known. Using this assumption, along with the air's axial velocity and the rotor velocity, the relative air angles can be determined. Also, the velocity of the air can be determined knowing these parameters. For more information relating to velocity diagrams refer to Cohen (Cohen, *et al.*, 1985).

Once the relative air angles and air velocities are calculated, the determination of design variables can be examined. Figure 4 shows characteristics of blades that are important to the design of a blade row.

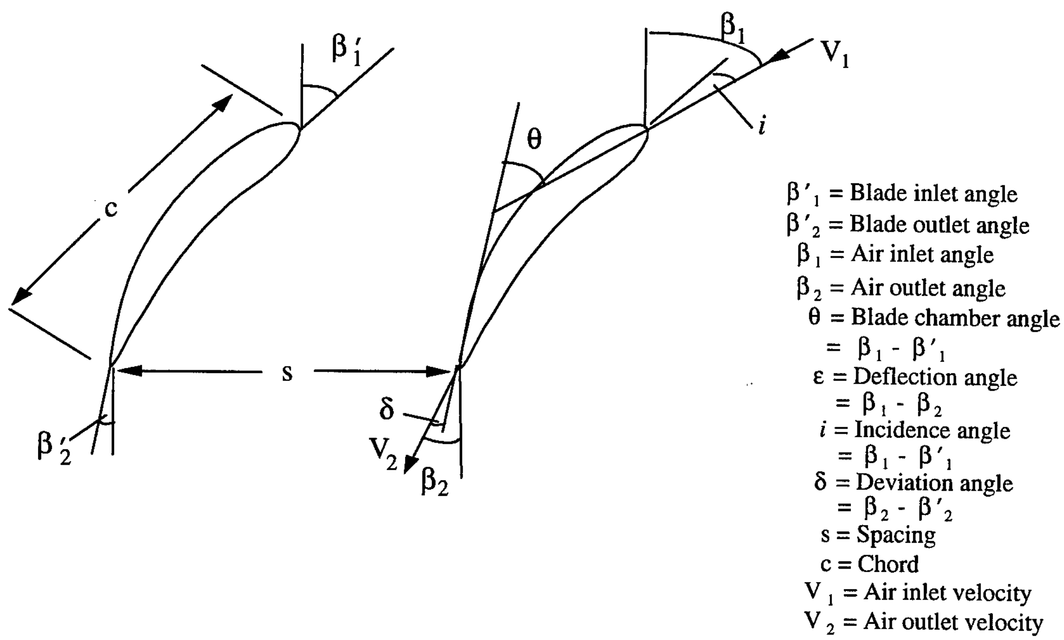

Figure 4. Blade geometry for a blade row.

Since the air angles and velocities are found from the velocity diagrams shown in Figure 3, the blade angles, spacing and chord of the blades become important variables. Not included in the figure is the length of each blade row from root to tip, as well as the velocity the blades are traveling. These affect how efficiently the blade row increases the pressure.

For a stage to increase the pressure in the most efficient way, design variables should be chosen to minimize the losses across the blade rows. By minimizing the losses, the maximum efficiency for the given operating conditions is also achieved. The losses can be identified by two quantities: the *lift* and *profile drag* coefficients, C_L and C_{DP}, respectively. Figure 5 shows the lift and drag forces in the blade row.

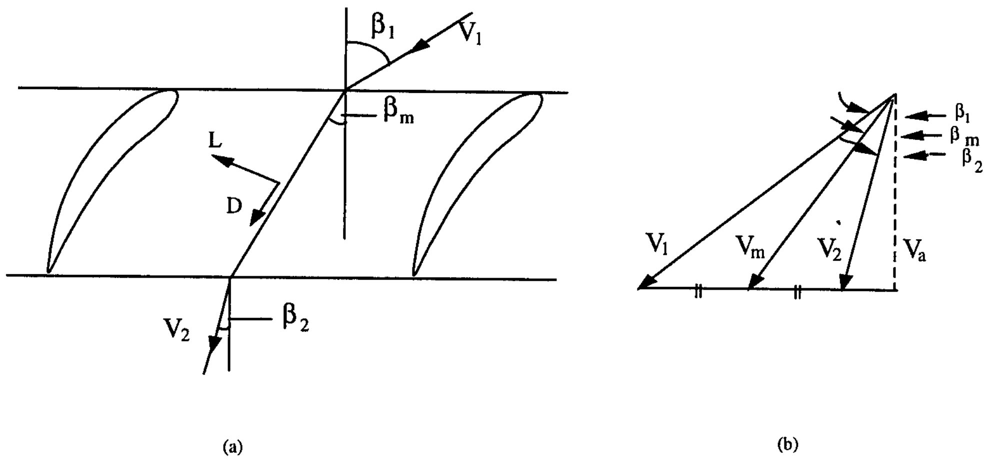

Figure 5. a)Losses in blade row. b)Relative velocity and air angles.

The velocities V_1 and V_2 in Figure 5b represent the velocity of the air relative to the rotor at the inlet and outlet; V_m is the mean velocity through the rotor. This is found to be important in testing compressor blade rows by use of cascade tunnels. Cascade tunnels (Lieblin, 1965) were developed to give experimental results from a cascade, or row of blades, based on the design considerations chosen for the blade. Using the results from cascade tests, equations were developed to determine approximations for the lift and drag coefficients. For the coefficient of lift losses, experimental results from cascade testing were plotted (Cohen, *et al.*,1985). From this information, an equation was derived for the coefficient of lift:

$$C_L = 2\left(\frac{s}{c}\right)(\tan\beta_1 - \tan\beta_2)\cos\beta_m \tag{1}$$

where

$$\beta_m = \tan^{-1}\left[\frac{1}{2}(\tan\beta_1 + \tan\beta_2)\right] . \tag{2}$$

As seen in Figure 6 and Equation 1, the coefficient of lift is affected not only by the air angles, but also by the spacing-to-chord ratio, s/c. This is one of the decision variables considered for the compressor design.

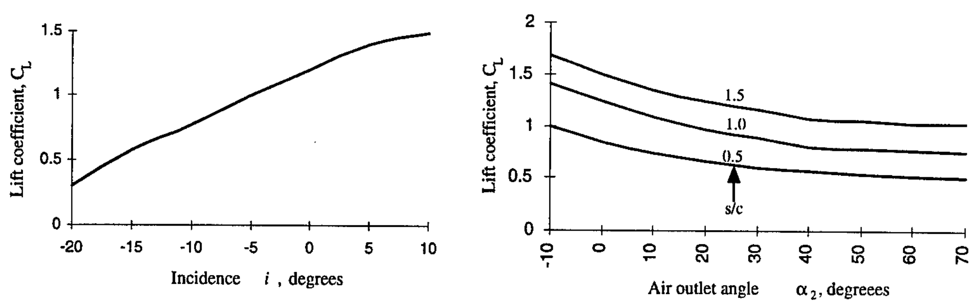

Figure 6. a) C_L given incidence angle. b) C_L for s/c ratios and outlet angles.

To determine the overall coefficient of drag, two considerations need to be taken into account other than the profile drag. One is additional drag losses due to the boundary layers at the compressor annulus walls, otherwise known as *annulus drag*. The other is losses from trailing vortices coming off the blades and losses from boundary layers between the tip of the blade and the annulus of the compressor, known as *secondary losses* (Cohen, *et al.*, 1985). The overall value of the coefficient of drag is given as the sum of the three drag forces: the coefficient of profile drag, the annulus coefficient, and the secondary loss coefficient; thus

$$C_D = C_{DP} + C_{DA} + C_{DS} \, . \tag{3}$$

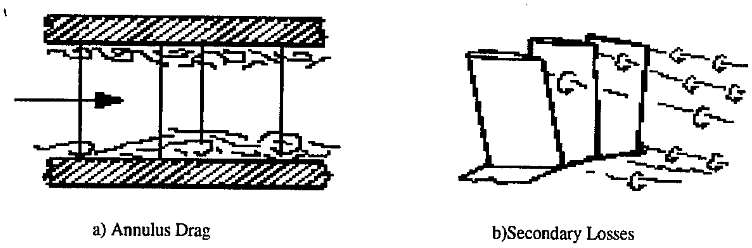

Figure 7. a) Annulus losses in compressor. b) Secondary losses in compressor.

The profile drag, based on cascade testing, is affected by the incidence angle as shown in Figure 8.

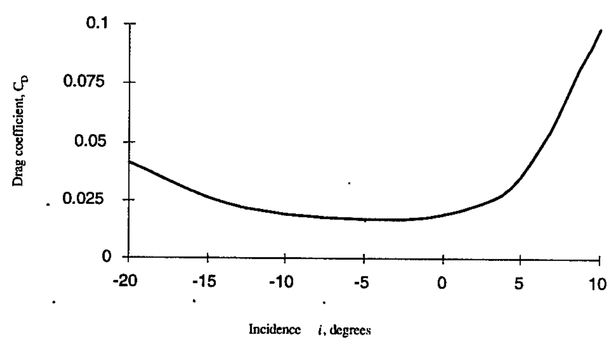

Figure 8. Drag coefficient for cascade of fixed geometry.

To minimize the losses, equations for the annulus and secondary losses were also developed based on the results of cascade tunnel testing. The coefficient of drag for the annulus, was determined to be

$$C_{DA} = 0.02\left(\frac{s}{h}\right). \tag{4}$$

For the secondary losses, the lift coefficient was determined to be influencing the secondary losses based on the relation

$$C_{DS} = 0.018C_L^2. \tag{5}$$

The lift and drag losses, otherwise known as the incompressible losses, can than be found by using the following equation showing the value of the losses:

$$\frac{\omega}{\frac{1}{2}\rho V_1^2} = \frac{C_D}{\frac{s}{c}}\left(\frac{\cos^2\beta_1}{\cos^3\beta_m}\right). \tag{6}$$

For transonic and supersonic flow, the additional loss of the shock wave has to be taken into account to properly model the compressor stage. Previous research (Miller, *et al.*, 1961) shows a way to get an approximation for the shock losses. The research assumed a shock pattern as shown in Figure 9.

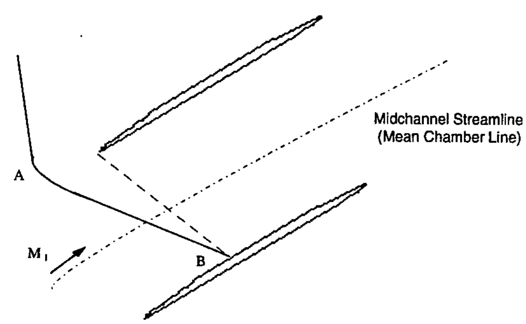

Figure 9. Mach number location to determine shock losses.

Using this approach, the shock loss can be approximated by the average of the Mach numbers at points A and B. The Mach number at A was assumed to be the relative Mach number at inlet. The value of the Mach number at B would be higher due to the turning of the blade suction side. Figure 10 shows how to approximate the shock losses based on the Mach numbers at A and B, as well as the supersonic turning angle($\Delta\theta$). The supersonic turning angle can be approximated by 0.5ϵ,

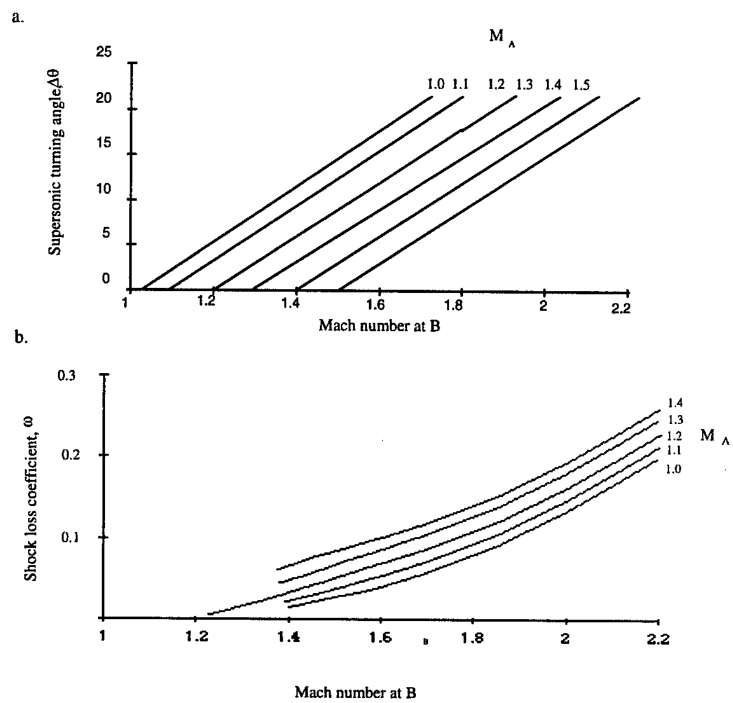

Figure 10. Determination of shock loss coefficient.

remembering that ϵ is the deflection of the blade. Figure 10a was used to obtain an equation for the shock losses based on the supersonic turning angle and the Mach number at A. First, a regression line was fit to find the supersonic turning angle given the Mach number at B when the Mach number at A is 1.0:

$$\Delta\theta = 28.57M_B - 30\,. \tag{7}$$

This does not take into account changes in the Mach number at A, so a correction factor was included to account for the Mach number at A. Once that was done, the equation was solved for the Mach number at B:

$$M_B = \frac{\Delta\theta + 20M_A + 10}{28.57}. \tag{8}$$

Figure 10b was then used to get a regression equation for the shock losses given a specific Mach number at B when the Mach number at A is 1.0. Once again a correction factor was included for the Mach number at A to account for possible changes in the Mach number at A

$$\omega_{shk} = 0.156M_B^2 - 0.334M_B + 0.2\,(M_A - 1)\,(M_B - 1) + 0.177\,. \tag{9}$$

Using Equation 8 that relates the Mach number at A and the supersonic turning angle to the Mach number at B, and substituting it for the Mach number at B in Equation 9, yields the shock loss coefficient equation:

$$\begin{aligned} \omega_{shk} &= 0.156\left(\frac{\Delta\theta\circ + 20M_A + 10}{28.57}\right)^2 - 0.334\left(\frac{\Delta\theta\circ + 20M_A + 10}{28.57}\right) \\ &\quad + 0.2\,(M_A - 1)\left(\frac{\Delta\theta\circ + 20M_A + 10}{28.57} - 1\right) + 0.177\,. \end{aligned} \tag{10}$$

The approximation of the shock losses can be added to the lift and drag losses to find the overall losses through the blade. This gives the total losses over the entire blade for the selected design. The efficiency for the blade can then be determined by comparing the actual pressure rise to the theoretical pressure rise. The actual pressure rise can be found by

$$\frac{\Delta p_{th}}{\frac{1}{2}\rho V_1^2} - \left(\frac{\omega}{\frac{1}{2}\rho V_1^2} + \omega_{shk}\right) \tag{11}$$

where the theoretical pressure rise can be determined by calculating

$$\frac{\Delta p_{th}}{\frac{1}{2}\rho V_1^2} = 1 - \frac{\cos^2\beta_1}{\cos^2\beta_2}. \tag{12}$$

The calculation for the blade efficiency based on the preceding discussion is:

$$\eta_b = 1 - \frac{\left(\frac{\omega}{\frac{1}{2}\rho V_1^2} + \omega_{shk}\right)}{\frac{\Delta p_{th}}{\frac{1}{2}\rho V_1^2}}. \tag{13}$$

To determine the efficiency over the stage, the rotor and stator efficiency are both calculated for given operating conditions. They are then combined to approximate the stage efficiency.

These are the fundamental concepts in determining the efficiency for a stage if the design variables and parameters of the compressor stage are determined. A more detailed introduction and proofs of these concepts are found in Cohen (Cohen, *et al.*, 1985) or Lieblin (Lieblin, 1965).

2.2.2 Off-Design

As stated, the compressor has to be capable of operating at a satisfactory level over a wide range of rotational speeds and inlet conditions for a specific design. This is because the compressor has to work under conditions other than the "ideal" that it was specifically designed for. Changes in operating conditions lead to changes of air angles and relative air velocities. These changes increase the incidence angle at the inlet to the rotor and stalling results when the incidence angle increases beyond a certain threshold. Once the geometry of the blade is determined, values for the incidence, deviation, and mean camber angle can be found (refer to Figure 4 for location of these angles).

The deviation angle (Cohen, *et al.*, 1985) for on-design can be approximated as

$$\delta_2 \approx 0.19\theta \,. \tag{14}$$

Using this deviation angle and the outlet air angle, the outlet blade angle can be determined. From Figure 11, Howell's correlation (Horlock, 1958) can be used to approximate the deflection. *Nominal*

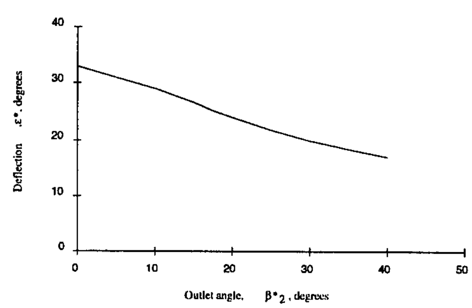

Figure 11. Deflection as a function of outlet angle on nominal conditions.

conditions refer to conditions pertaining to a cascade which is 80 percent of its maximum "stalling" deflection (Horlock, 1958). Now the blade inlet angle (i^*) can be found since the deflection and outlet blade angle are known. The off-design performance can be examined at differing values of incidence based on

$$\frac{i - i^*}{\epsilon^*}. \tag{15}$$

The incidence can be varied to achieve different values for this equation. New air angles can be determined based on the incidence. The efficiency of the stage can be calculated by keeping the same design variables, but having the air angles change. For more information and detail refer to Horlock (Horlock, 1973).

2.3 Compressor Model

Now that a basic introduction of compressor theory has been introduced, the model of the two-stage compressor can be developed. First, the assumptions used in the development of the model are discussed. Then, the constraints representing the aerodynamic conditions; including constraints for the air angles, velocity, and losses are shown. Finally, the objective function of maximizing the efficiency for the two-stage compressor is developed.

2.3.1 Operating Condition Assumptions

As stated in the compressor theory section, some assumptions must be made to model the compressor in two stages as a steady-state system. Two basic assumptions made to simplify the model are: flow is incompressible across each stage, and the axial-flow velocity of the air is constant. Incompressible flow can be assumed due to the small pressure increases over each stage. The annulus flow area is decreased along the compressor to maintain a constant axial-flow velocity. Therefore, these assumptions are reasonable and can be made to help simplify the model. Another assumption associated with the small pressure changes is the assumption of constant specific heats. This allows the power input to the compressor to be based only on the change in temperature over the compressor. These assumptions are made to simplify the model to develop equations to determine the blade geometry of each blade row.

To develop a numerical model using these assumptions, inlet conditions to the compressor must to be defined. Also, some consideration to the values of certain blade aspects are necessary to develop constraints for the stage. Inlet conditions include the temperature, pressure, density, and velocity of the air just as it enters the first stage of the compressor. Blade aspects need to be taken into account to determine the air angles and the relative velocities of the air. These aspects are the ratio of the blade radius at the hub of the blade to the radius of the blade at the tip, and the diffusion factor of the first stage. Table 1 shows the inlet conditions as well as the blade aspects that were used in this research.

Table 1. On-design operating conditions

Inlet operating conditions	
Axial air velocity, $C_a(\frac{m}{s}) \equiv$	150
Stagnation temperature, T_{01} (K) $\equiv$	288
Stagnation pressure, P_{01} (bars) $\equiv$	1.01
Stage 1 pressure ratio, $\frac{P_{03}}{P_{01}} \equiv$	1.25
Stage 2 pressure ratio, $\frac{P_{05}}{P_{03}} \equiv$	1.25
Stage 1 rotor tip height, $r_{rt}^1 \equiv$	0.2262
Stage 1 stator tip height, $r_{st}^1 \equiv$	0.2258
Stage 2 rotor tip height, $r_{rt}^2 \equiv$	0.2258
Stage 2 stator tip height, $r_{st}^2 \equiv$	0.2255

Also, the absolute air angles into each stage for the operating conditions can be set by the designer. This research examines the compressor design at differing values of the inlet absolute air angles to each stage, to determine if there is an optimal angle that optimizes the efficiency of the compressor.

2.3.2 Compressor Constraints

Once the values in Table 2 were defined, the constraints for the model could be developed, taking into account the first two steps used to design a compressor. Upper and lower bound constraints on the design variables were included to ensure the ranges of the variables chosen allow the compressor to be operational for the on-design of the compressor (values for the upper and lower bounds are found in Table 3).

2.3.2.1 Rotor Design Constraints

An initial step for compressor design is determining the relative air angles and velocity of the air at the tip and mean radius of the rotor. The information at the blade tip is used to calculate shock losses for the blade, while the mean radius blade information is used to calculate the efficiency of the rotor. Since the air angles, velocities, and losses are dependent on the values of the inlet conditions and the rotational speed of the rotor, only the equations used to find these parameters are shown. The values of the rotational velocity of the rotor change to determine how the rotor velocity affects the efficiency of the compressor. To determine the relative velocity (V) at each location of the blade, the rotational velocity of the blade (U) at each location has to be determined:

$$U_t = 2\pi r_{tr} N \tag{16}$$

$$U_m = \pi \left(1 + \frac{r_h}{r_t}\right) r_t N. \tag{17}$$

Once the rotational velocity of the blade at each location along the blade is determined, the relative velocity of the air can be calculated as,

$$V_{1t} = \sqrt{(u_t)^2 + (C_a)^2} \tag{18}$$

$$V_{1m} = \sqrt{(u_m)^2 + (C_a)^2}. \tag{19}$$

The Mach number at the tip can be determined using relative velocity and the speed of sound for the conditions at the blade tip. The speed of sound can be calculated by

$$a = \sqrt{\gamma R T} \tag{20}$$

yielding,

$$M_{1t} = \frac{V_{1t}}{a}. \tag{21}$$

If the Mach number at the blade tip is supersonic, then shock losses need to be included in the overall losses across the blade. If the Mach number is subsonic, there are no shock losses to account for in the overall blade losses.

Having the blade rotational velocity, inlet absolute air angle and axial flow velocity, the relative air angles at the tip and mean blade radius can be found:

$$\beta_{1t} = \arctan\left(\frac{U_{1t} - C_{w1}}{C_a}\right) \tag{22}$$

$$\beta_{1m} = \arctan\left(\frac{U_{1m} - C_{w1}}{C_a}\right) \tag{23}$$

where $C_{w1} = C_a \tan \alpha_1$.

To determine the relative air angles at station 2, the change in the whirl velocity component must be found. Using the assumption of constant specific heat, the power input to the compressor stage can be given as

$$W = m c_p \Delta T_o \tag{24}$$

where ΔT_o is the change in stagnation temperature across the blade. By considering the change in angular momentum of the air passing through the rotor, the power input can also be calculated as

$$W = m U \Delta C_w . \tag{25}$$

These two equations can then be used to solve for the change in tangential velocity across the rotor. The tangential or whirl velocity is calculated by

$$\Delta C_w = \frac{c_p \Delta T_o}{\lambda U} \tag{26}$$

where λ is known as the work-done factor.

The work-done factor is introduced to account for changes in the boundary layer through the compressor. As the air flows through the compressor, the boundary layers from the annulus wall and blade tip clearance increase in area. This increase results in a reduction of work capacity as the number of stages the air travels through increases. The work-done factor accounts for this reduction in work capacity. Approximations of λ have been determined from experimental results. Figure 12 shows how the work done factor, λ, is affected by the number of stages in the compressor.

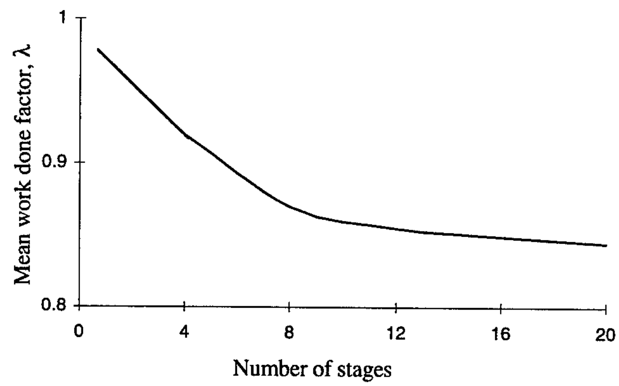

Figure 12. Variation of work done factor for number of stages.

Using Equation 26, the whirl velocity at the tip and mean radius of the outlet of the rotor blade can now be determined. Using the velocity diagrams from Figure 3 and the whirl velocity at the outlet of the rotor, the relative air angles at station 2 can be found:

$$\beta_{2t} = \arctan\left(\frac{U_t - C_{w2t}}{C_a}\right) \tag{27}$$

$$\beta_{2m} = \arctan\left(\frac{U_m - C_{w2m}}{C_a}\right). \tag{28}$$

Now that the inlet and outlet angles have been determined, the relative velocity at the rotor outlet can be calculated. The velocity at the rotor outlet can be found by:

$$V_{2t} = C_a \cos\beta_{2t} \tag{29}$$

$$V_{2m} = C_a \cos\beta_{2m}. \tag{30}$$

This change in velocity across the rotor affects the development of a boundary layer over the length of the blade. Boundary layers increase where there are high velocity gradients. The diffusion factor is used to account for the losses due to the development of these boundary layers. The diffusion factor is based on the development of velocity gradients across a blade row. Figure 13 shows incompressible losses at regions of the blade for different values of the diffusion factor. The diffusion

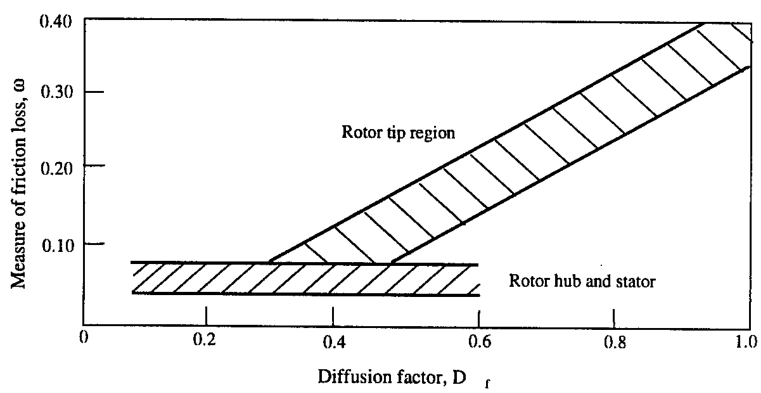

Figure 13. Losses for given diffusion factors.

factor can be determined by the change in velocity and the s/c ratio by the equation

$$D_f = 1 - \frac{V_2}{V_1} + \frac{\Delta C_w}{2V_1} \cdot \frac{s}{c}. \tag{31}$$

For the on-design of the two-stage compressor in this research, a diffusion factor of 0.5 at the mean blade radius is used as a upper bound for the diffusion factor. By using this upper limit of the diffusion factor, the change in velocity is limited. This also limits the losses that can occur over the blade.

Since the relative air angles have been calculated for the given inlet conditions, the losses over the rotor can now be found. These losses include the friction losses over the blade and the possible shock losses for supersonic flow at the tip of the rotor.

2.3.2.2 Shock Loss Constraints

Now that conditions at the inlet and outlet of the rotor are established, the losses over the rotor can be determined. The inlet and outlet relative air angles at the tip are used to calculate the supersonic turning angle to help determine the losses associated with the possible shock waves that occur in the compressor. For a three-dimensional model of the compressor stage, the shock loss over the entire length of the blades would have to be taken into account. This research uses the shock losses at the blade tip to be an initial estimate of the total shock losses that can occur over the entire length of the blade.

Referring to section 2.2.1, the equations used to calculate shock losses for given air angles are shown. The supersonic turning angle based on the change in inlet and outlet angles at the blade tip is found to be $\frac{1}{2}(\beta_{1t} - \beta_{2t})$. This angle can be used in association with the relative Mach number at the inlet rotor blade tip to determine if there are shock losses across the blade. If the Mach number shows the air is supersonic, shock losses occur over the rotor. Referring to Equation 10, the shock losses can be found by

$$\begin{aligned} \omega_{shk} &= 0.156\left(\frac{\Delta\theta\circ+20M_A+10}{28.57}\right)^2 - 0.334\left(\frac{\Delta\theta\circ+20M_A+10}{28.57}\right) \\ &+0.2\left(M_A-1\right)\left(\frac{\Delta\theta\circ+20M_A+10}{28.57}-1\right)+0.177. \end{aligned} \tag{32}$$

In addition to the shock losses, the rotor incurs incompressible losses due to the lift and drag of the air as it flows through the rotor.

2.3.2.3 *Incompressible Loss Constraints*

The angles at the mean blade radius are used to determine the average incompressible losses for the rotor. This is a preliminary value of the incompressible losses for the rotor. As with the shock losses, for a three-dimensional model of the flow, the incompressible losses would need to be examined at different radii along the blade. For this research, only the losses at the mean blade radius are used to determine the incompressible losses over the rotor. Referring back to 2.2.1, there are two types of incompressible losses: lift and drag. They are calculated by using equations for the coefficient of lift and drag, C_L and C_D. Equation 1 gives the formula to calculate the lift coefficients.

The coefficient of drag is calculated by

$$C_D = 0.018 + 0.018C_L^2 + 0.02\frac{s}{h}. \tag{33}$$

The value of 0.018 in equation 33 represents the profile drag loss. It is assumed to be a constant since it is based on the incidence angle as shown in Figure 14. For this model, the incidence angle is assumed to be zero. Figure 14 shows the profile drag loss is not very sensitive to the change of the incidence angle in the region of zero incidence. Using the coefficient of drag, the incompressible

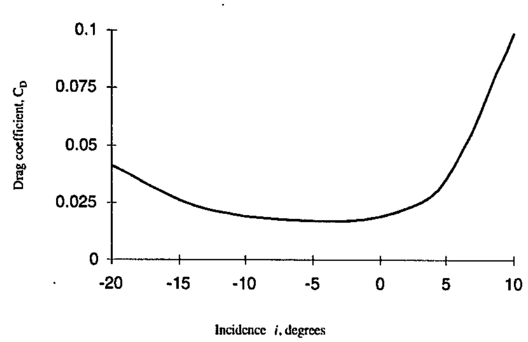

Figure 14. C_{Dp} for fixed blade geometry.

losses can be determined by the equation

$$\frac{\omega}{\frac{1}{2}\rho V_1^2} = \frac{C_D}{\frac{s}{c}}\left(\frac{\cos^2\beta_1}{\cos^3\beta_m}\right). \tag{34}$$

These losses can be added to the value of the shock losses to determine the overall losses in the rotor

$$\omega_{\mathrm{T}r} = \frac{\omega}{\frac{1}{2}\rho V_1^2} + \omega_{shk}. \tag{35}$$

Once the overall losses are found, the actual pressure rise (Δ P_{act}) over the rotor can be calculated by subtracting the overall losses from the theoretical pressure rise. The theoretical pressure rise is the possible increase in pressure the rotor can achieve for the given operating conditions. It can be calculated in terms of the inlet dynamic head and the air angles by

$$\Delta P_{th} = 1 - \frac{\cos^2\beta_1}{\cos^2\beta_2}. \tag{36}$$

The efficiency of the rotor can now be found by taking the ratio of the actual pressure rise over the theoretical pressure rise,

$$\frac{(\Delta P_{act})_r}{(\Delta P_{th})_r}. \tag{37}$$

2.3.2.4 Stator Design Constraints

To get the efficiency of the entire stage, the efficiency of the stator must be calculated. This is calculated in the same fashion as the efficiency for the rotor. First, the air angles and velocities are found, then the losses across the stator are calculated. Finally, the actual and theoretical pressure increases across the stator can be found in order to calculate the efficiency of the stator.

Finding the air angles of the stator is much simpler than it was for the rotor. The inlet angle to the stator is equal to the outlet air angle of the rotor. For the outlet air angle of the stator, the stator can be designed to give any outlet angle desired. It is important to remember that the outlet air angle of the stator is the inlet air angle to the next stage. This is a factor in determining the overall efficiency of the two-stage compressor. Chapter 4 gives the results of how changes in the outlet angle of the stator affects overall compressor efficiency. Since the inlet and outlet air angles can be determined, the actual and theoretical pressure increases can be calculated. The efficiency of the stator is calculated as the ratio of the actual pressure rise over the theoretical pressure rise. An equation of the stator efficiency is needed, but first the losses over the stator need to be found to determine the actual pressure increase. The losses over the stator are discussed in the next section.

2.3.2.5 Stator Loss Constraints

Since the stator is not moving, the relative and absolute conditions are equal. Shock losses do not need to be considered since the absolute velocity through the stator will be less than supersonic conditions. Therefore, only the incompressible losses need to be considered across the stator. The incompressible losses in the stator are found the same way they were in the rotor. It was found that the coefficients of lift and drag could be determined from the change in whirl velocity across the blade row. This being the case, the mean air angle is given as

$$\alpha_{ms} = \arctan\left(\frac{C_{w2} + C_{w3}}{2C_a}\right). \tag{38}$$

The mean air angle can be used to calculate the value of the coefficient of lift for the stator. The formula for C_L in the stator is given by

$$C_L = 2\left(\frac{s}{c}\right)\left(\frac{C_{w2} - C_{w3}}{C_a}\right)\cos\alpha_m. \tag{39}$$

The value of the lift coefficient for the stator can be used with equation 33 to determine the value of the drag coefficient for the stator. The s/h ratio will be for the stator as well. A value for the total losses over the stator can be determined by using C_L from Equation 39, the s/c ratio of the stator, and Equation 34. To determine the efficiency of the stator, the theoretical pressure increase can be calculated using Equation 36 and the relative angles at the outlet and inlet of the stator, just as was done in the rotor.

2.3.2.6 *Stage Efficiency*

Now that the efficiency of both the rotor and the stator can be determined, the overall stage efficiency can be approximated. It is a combination of both the efficiency of the rotor and the stator. The equation for the stage efficiency based on the rotor and stator efficiency is

$$\eta_{st} = \Lambda\eta_r + (1 - \Lambda)\eta_s. \tag{40}$$

The parameter Λ is known as the *degree of reaction* - the ratio of the pressure rise in the rotor over the pressure rise of the stage, $\left(\frac{\Delta P_r}{\Delta P_r + \Delta P_s}\right)$. By changing the air angles at the inlet and outlet of the stages, this research attempts to determine what degree of reaction each stage yields for the overall optimal efficiency of the two-stage compressor.

2.3.2.7 *Successive Stages*

To determine the efficiency of successive stages, the same approach is taken as for the stage presented in the previous section. The inlet conditions for successive stages are the outlet conditions from the previous stage. Therefore, the outlet conditions for the previous stage need to be determined. The inlet air angle, α, to each stage is chosen by the designer, so it is independent of the previous stage. Also, there are changes to the pressure and temperature from the inlet conditions of the previous stage to the inlet conditions of the next stage. This is due to the air undergoing changes in the previous stage. The stagnation pressure increases by the pressure ratio across the previous stage:

$$P_{03} = P_{01} \cdot \frac{P_{03}}{P_{01}}. \tag{41}$$

The stagnation temperature increase is based on the polytropic efficiency of the previous stage and can be calculated by

$$T_{03} = T_{02} = T_{01}\left[\frac{P_{02}}{P_{01}}\right]^{\frac{1}{\eta_p}\left(\frac{\gamma-1}{\gamma}\right)}. \tag{42}$$

The polytropic efficiency for this research was defined as 0.9. As the pressure ratio approaches 1, the stage efficiency approaches the value of the polytropic efficiency. Compressors are designed to achieve an on-design efficiency in the range of 90%; therefore, the use of 0.9 for the polytropic efficiency is an appropriate choice. Once the inlet conditions are determined, the efficiency can be determined for the stage.

2.3.3 Bounding Constraints

For the specific problem of the two-stage compressor analyzed in this research, bounds are placed on inlet conditions that are being treated as variables in this research. The bounding constraints used for this problem are shown in Table 2.

Table 2. Upper and Lower bounds for decision variables

$-20 \leq \alpha_1 \leq 20$	$0.4 \leq \left(\frac{s}{c}\right)_r^1 \leq 2$	$\left(\frac{r_h}{r_t}\right)^1 = 0.5$
$-20 \leq \alpha_3 \leq 20$	$0.4 \leq \left(\frac{s}{c}\right)_s^1 \leq 2$	$\left(\frac{r_h}{r_t}\right)^2 = 0.5$
$\alpha_5 = 0$	$0.4 \leq \left(\frac{s}{c}\right)_r^2 \leq 2$	$\left(\frac{h}{c}\right)^1 \leq 4$
$150 \leq N \leq 250$	$0.4 \leq \left(\frac{s}{c}\right)_s^2 \leq 2$	$\left(\frac{h}{c}\right)^2 \leq 4$
$D_f^1 \leq 0.5$	$D_f^2 \leq 0.5$	

The bounds on α_1 and α_3 are set to examine the possible angles while maintaining the design as operationally feasible. The stage 2 outlet angle, α_5 is set to zero since the air leaves the compressor and enters the combustion chamber at this point. This forces the air leaving the two-stage compressor to travel in the direction of the axial-flow velocity vector. This was done since the air would enter the combustion chamber at this point. The effect of any change in the outlet air conditions from the compressor on the performance of the overall engine is not known. Analysis should be performed on the combustion chamber to determine what affect changes to α_5 would have on engine performance. This limits the flexibility the model has of representing the compressor, but analysis of the combustion chamber should be included before changes are made to the outlet conditions of the compressor. The rotational velocity (N) of the rotor blade is bounded to keep the material stresses of the blades at acceptable levels. The upper bound placed on the diffusion factor, D_f, are to limit the losses in the rotor. Refer to Figure 13 to see the effect the diffusion factor has on the losses in the rotor.

The bounds placed on the spacing-to-chord (s/c) ratios are set to maintain operational feasibility at the extremes. The equality constraints for the ratio of the blade radius at the hub and tip for each stage were set to annulus flow area. The blade height-to-chord ratio is also set to simplify the model. These constraints limit the model by setting an upper bound on the values for these parameters.

2.3.4 Objective Function

As stated in the problem statement, the objective function used in the two-stage compressor model is to optimize the efficiency of the two-stage compressor. The efficiency of two stages can be calculated as the sum of the actual pressure rise of each stage divided by the sum of the theoretical pressure rises over each stage:

$$\eta_{\mathrm{T}}^{12} = \frac{\Delta P_{act}^{1} + \Delta P_{act}^{2}}{\Delta P_{th}^{1} + \Delta P_{th}^{2}} . \tag{43}$$

The objective function can be used with the constraints to determine the optimal geometry of the blades. This geometry includes the s/c ratio for each blade row, the inlet and outlet air angles for each stage, and the rotational velocity of the rotors in the compressor.

2.3.5 Model Validation

Once the model was developed, it is important to validate the model to ensure it properly represents the compressor. Two steps were taken to validate the model used in this research.

First, the operating conditions considered for the model were the same conditions that were used in a example from Cohen (Cohen, *et al.*, 1985). In the example, specific values for the rotational velocity, spacing-to-chord ratios, and air angles were used just to show how the stage efficiency was determined. These values were put into the model to ensure the model returned the same efficiency as in Cohen.

Second, The results from the model were compared to results from Massardo and Satta (Massardo and Satta, 1990) to determine if both models behaved in the same manner when variables where changed, respectively. Massardo and Satta's results were compared to those of the first stage of the compressor model developed in this research as the values of the inlet and outlet air angles varied for validation. Both models showed the efficiency for a single stage increases as the outlet angle, α_3, is increased from 0 to 20 degrees. Both models also showed the stage 1 efficiency decreased slightly as the inlet air angle, α_1, increased from 0 to 20 degrees. The exact values of the efficiency for each model were not the same, but that was expected since the operating conditions were different for each model. This comparison shows general patterns that help validate the model developed in Chapter 2 is an accurate representation of a two-stage compressor.

2.4 Conclusion

The basic theory that is considered when developing a model representing a compressor has been introduced in this chapter. It included aspects relating to calculating the incompressible and shock losses and using those losses, along with the pressure rise across the blade row, to calculate the efficiency of each stage.

Also included in this chapter was the application of modeling a two stage compressor used in this research. The aerodynamic constraints of the model were developed and presented. A discussion of the bounding constraints for the decision variables and the design parameters followed to set an operating range for the compressor. Finally, the objective function - which is to maximize the efficiency of the two stage compressor - is discussed. The next chapter introduces and develops the necessary optimization approaches used to find an optimal compressor design.

Chapter 3 - Optimizing the Compressor Model

3.1 Introduction

This chapter focuses on the optimization methods available for use in solving the compressor model developed in Chapter 2. The first section of the chapter introduces the concept of optimization and methods that may be used for the compressor design model. After the methods have been introduced and the reader has an idea of the various optimization methods that are available, a more in-depth discussion of the methods used to determine an optimal compressor design is presented.

Once the background for the different optimization methods used in this research are discussed, the balance of the chapter focuses on the implementation of the optimization methods to the compressor model. Examination of single stage, two-stage and N-stage compressors are presented.

3.2 Optimization Literature

Optimization can be defined as obtaining the most desired result of an objective for a given problem. The *objective* can be to either minimize a cost associated with the problem, or maximize a positive benefit. The objective is a function of the decision variables pertaining to the problem; therefore, optimization can be thought of as the determination of variables that yield the most desired result of this objective function. Often, constraints are placed on the problem; these constraints limit the values of the decision variables.

There are many methods available to determine the optimal value of the objective function based on the model of the problem. To narrow the scope of this literature review, only the methods that may be used to solve the compressor design problem are discussed. Two important aspects of the compressor model are used to narrow the types of methods considered. First, the compressor model has a nonlinear objective function and aerodynamic constraints that are nonlinear, so only nonlinear optimization methods are considered. Also, the compressor model has the unique feature in that it can be separated into individual stages. This allows for the possibility of the compressor model to be solved by stage-wise methods. Both the nonlinear and stage-wise methods used in this research are presented to introduce the reader to the basic concepts of the optimization methods considered for use in optimal compressor design.

3.2.1 Nonlinear Methods

There are many methods that can be used to solve problems containing nonlinear objective functions and/or nonlinear constraints. The table below shows some of the more common optimization methods available for solving nonlinear programs. Not all of these methods are discussed in this thesis; only the methods used in the application of determining an optimal design for the compressor are discussed. For more information concerning these methods, refer to Rao (Rao, 1996) or Bazaraa (Bazaraa, *et al.*, 1979).

Table 3. Nonlinear optimization methods

Direct methods	Indirect methods
Random search methods	Sequential unconstrained methods
Objective and constraint approximation methods	Interior point function method
Sequential linear programming methods	Exterior point function method
Sequential quadratic programming methods	Augmented lagrangian multipier method
Methods of feasible directions	
Zoutendijk's method	
Rosen's gradient projection method	
Generalized reduced gradient methods	
Stage-wise methods	
Dynamic programming	

The stage-wise and Generalized Reduced Gradient (GRG) methods were the two methods used to solve for an optimal compressor design, so a brief introduction of both methods is presented. This is meant to be an introduction to the two optimization methods; a discussion of why both of these methods were chosen can be found in section 3.2.

3.2.2 Multistage Optimization

Multistage problems can be distinguished by their ability to be separated into individual stages. To examine a multistage problem, it is beneficial to first look at the process for a single stage. The model for a single-stage decision problem can be represented as shown in Figure 15. The optimization of the single stage is based on the input parameters (S) of the problem and the choice of decision variables (X). Depending on the values of S and X, the output (T) of the problem will be affected by a stage transformation. How the input parameters and decision variables effect the

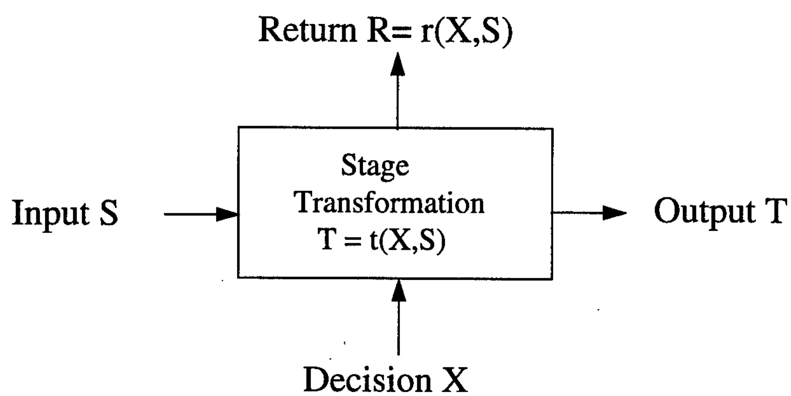

Figure 15. Single stage decision problem.

system can be determined by the objective function (R). This concept can be used when analyzing a multistage problem. Multistage problems are linked together in such a way that the output of one stage becomes the input to the next stage. This is represented in Figure 16

The objective function of each stage is used in the process of determining the problem's total objective function. It is defined as finding $x_1, x_2, ..., x_n$ that satisfies the function representing the overall decision based on the objective functions of each individual stage, $f(R_1, R_2, ..., R_n)$. To use this function, the objective function of each stage must depend only on the input parameters and decision variables for that stage. This concept is known as the *separability* of the objective function. This is because the objective function is represented as a *composition* of the individual stage returns. It allows each stage to be evaluated separately and the individual objective functions are combined to determine the overall objective function. The overall system's objective function

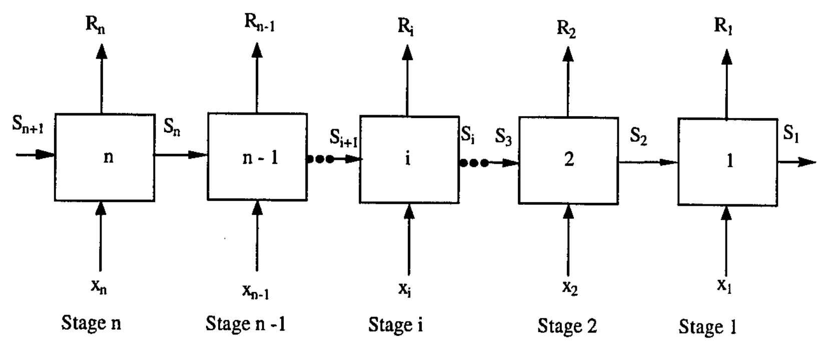

Figure 16. Multistage decision problem.

also has to maintain *monotonicity* with the return of each stage for all possible decision variables. The concept of monotonicity is shown in the following brief example. Given decision variables A and B, where an individual stage return for the decision variables is

$$R_i\left(x_i = A, S_{i+1}\right) \geq R_i\left(x_i = B, S_{i+1}\right) \tag{44}$$

The objective function is said to be monotonic if the following inequality is also satisfied:

$$f(x_n, x_{n-1}, \ldots, x_i = A, \ldots, x_2, x_1, S_{n+1}) \geq f(x_n, x_{n-1}, \ldots, x_i = B, \ldots, x_2, x_1, S_{n+1}) \tag{45}$$

Equation 45 has to hold for all $i = 1, 2, \ldots, n$. Both separability and monotonicity are required to allow for a multistage problem to be solved in a stage-wise manner. When both hold, the problem can be decomposed into separate stages and the individual stage returns can be used in an overall objective function to determine the optimal design for the multistage problem.

To optimize the problem when the output of one stage is the input to the following stage, the *principle of optimality* is considered. The principle of optimality can be thought of as: an overall optimal design is also the optimal design for the following stages given the output from the previous stage. A recurrence relationship can be developed to solve for the optimal design. Referring to Figure 16, using the principle of optimality to create a recurrence relationship, the optimal objective function can be determined by

$$f_i^*(S_{i+1}) = \underset{x_i}{opt} \left[R_i(x_i, S_{i+1}) + f_{i-1}^*(S_i)\right] \tag{46}$$

where $f_{i-1}^*(S_i)$ is the optimal objective function of the multistage problem having $i-1$ stages based on the input parameter S_i. The principle of optimality essentially states: by adding the optimal return of the ith stage for given input parameters (S_{i+1}) and decision variables (X_i) to the $(i-1)$st optimal objective function, it yields the optimal ith stage objective function. A recurrence relationship is developed for all multi-stage subproblems, from a single-stage problem to an $(i - 1)$ multistage problem. These recurrence relationships allow for determination of the optimal decision variables at each stage.

This was meant to be a brief introduction to multistage problems. The basic concepts of what was necessary, mainly *separability* and *monotonicity* of the objective function, to allow a multistage problem to be decomposed and solved using a stage-wise approach. For a more in-depth explanation, refer to Nemhauser (Nemhauser, 1966) or Denardo (Denardo, 1982).

3.2.3 Generalized Reduced Gradient Method

The generalized Reduced Gradient(GRG) method is used in the compressor model to determine the optimal design for a single stage. The following section is meant to present to fundamental background of the GRG method. It's application relating to the compressor model is discussed later in the chapter, specifically in section 3.2.2.

The GRG method is a feasible direction method. What makes it unique is how it solves for the feasible direction and establishes the step length to take in that feasible direction.

Slack variables are added to inequality constraints to transform them into equality constraints, including any upper and lower bound constraints on the decision variables. Doing this allows a basic and nonbasic constraint set to be determined from the set of all constraints. Using the basic and nonbasic constraint sets, the generalized reduced gradient can be computed. After this step, the generalized reduced gradient method tests the basic and nonbasic constraint sets for convergence. If they are within a specified tolerance, the design can be taken as the optimal solution. Otherwise, a search direction and step size need to be determined for the next selection of design variables.

3.3 Optimization Approaches

This section focuses on optimizing the compressor design based on the model developed in Chapter 2. Given the multi-stage aspect of the problem, an optimization approach that takes this structure into consideration, even exploiting it, is a reasonable approach to what may be a complex problem. In this research two approaches were taken: one as a single objective based on the composition of the stage pressure rises as seen in Equation43, and the other as a stage-wise optimization of cumulative returns, as in dynamic programming. The specific justification for methods of optimization employed are discussed.

3.3.1 Single Stage Optimization

First, it is convenient to examine the optimization method that would best solve a single-stage compressor. This allows for examination of the objective function and constraints to determine what characteristics are important to the model.

The mathematical model of a single stage compressor is developed in Chapter 2. It shows that the objective function and many of the constraints are nonlinear. This narrows the type of optimization methods appropriate for the model. The number of decision variables and the bounds placed on them generates a large feasible region. This large feasible region makes it difficult for the "random search" methods to efficiently examine the feasible region. Since the objective function and constraints are nonlinear, methods for objective and constraint approximation may be used. Linearizing the objective function and constraints, as done in sequential linear programming, would provide only gross approximations, losing information necessary to achieve the optimal design. Indirect methods such as penalty function methods would be difficult to apply to this problem since there are many constraints, and a penalty function has to be added to every constraint to form the unconstrained problem. The GRG method was chosen as the best alternative to solve this particular problem, since most of the constraints were equality constraints. Also, in practice, GRG-based methods have proven to be robust in the types of problems they solve.

Referring to section 3.2.3, the GRG method is based on the idea of eliminating variables using equality constraints. This method adds nonnegative slack variables to each inequality constraint to obtain a problem with all equality constraints. Since most of the constraints in the compressor model are already equality constraints, the GRG method can be easily applied to the model to solve for the optimal design.

There are many "commercial" solvers that use GRG-based methods to solve nonlinear models (Moré and Wright, 1993). The one chosen for use in this research is the Microsoft Excel Solver package. One reason the Excel solver was chosen over the other available solvers was its ability to determine the air angles and losses of the compressor blades by linking the spreadsheet cells. Also, it has the ability to evaluate the multiple trigonometric functions found in the constraints. The Excel Solver package uses a variant of the Generalized Reduced Gradient code (Microsoft, 1993).

For a single-stage compressor, the GRG method is considered to be an efficient method to use in solving for the optimal design. This is based on the number of equality constraints placed on the stage to determine the air angles and losses of the stage. For multiple stages, it is yet to be shown that the GRG method is still the best method to use. This question was taken into consideration as this research examined the compressor model for two stages of a compressor.

3.3.2 Two-Stage Compressor Optimization

When the second stage is added it increases the complexity of the compressor model. The outlet conditions of the first stage become more important to the efficiency of the compressor since they are the inlet conditions to the second stage. Also, the number of decision variables and the complexity of the model increase. Instead of determining the optimal air angles and spacing-to-chord ratios for a single stage, those variables need to be determined for two stages. The number of constraints also increases, adding to the complexity of the model. The GRG method can still be used to solve for the optimal design of a two-stage compressor, however, as more stages are added, the GRG method would not be capable of solving for the optimal compressor design due to growth in the number of design variables and constraints involved in a more general, multi-stage compressor. One way to deal with the case of multiple stages would be to break the stages apart and solve the subproblems to achieve an overall optimal design. This is where dynamic programming becomes an important method. As the number of stages increases, the use of dynamic programming becomes more useful than using the GRG directly on the composite method. The approach of using dynamic programming is considered for solving the two-stage compressor problem.

In solving for the optimal design of the two stage compressor, with consideration given to possibilities for future research, dynamic programming was chosen as the method to incorporate. The way it was incorporated into the two stage compressor model is presented.

3.3.3 Dynamic Programming Approach

Dynamic programming breaks the larger problem of the two-stage compressor model into two subproblems; each subproblem relating to one stage of the compressor. Recalling the literature review of dynamic programming in Chapter 3, a value for the overall objective function is based on the optimal objective function of each stage for given inlet conditions and decision variables of each stage. Also from the literature review, the overall objective function needs to have separability to apply dynamic programming methods. For the current objective function of the two stage compressor model given in equation 43, the efficiencies of both stages are not separable. Therefore, an approximation needs to be made that allows the efficiencies of both stages to be separated. The overall efficiency of the two stage compressor was found in Equation 43 to be

$$\eta_{\mathrm{T}} = \frac{\Delta P^1_{act} + \Delta P^2_{act}}{\Delta P^1_{th} + \Delta P^2_{th}} \,. \tag{47}$$

The approximation is based on the assumption that the losses for both stages are approximately equal, as well as assuming the theoretical pressure rises over each stage are approximately the same. Using these assumptions, the efficiency of each stage can be considered to be the same value. From this, the two-stage efficiency can be approximated by taking the average of the efficiency of the first and second stages:

$$\eta_{\mathrm{T}} = \frac{1}{2}\left(\eta^1 + \eta^2\right) \,. \tag{48}$$

Appendix A shows the difference in the actual efficiency compared to the approximation for changes in the efficiency of each stage. The largest difference between the actual efficiency and the approximation is 1.154%. This is taken to be a valid approximation of the overall efficiency of the two-stage compressor. However, this approximation needs to be reevaluated when multiple stages are added to determine how well it approximates the actual efficiency of the overall compressor.

This approximation of the objective function also allows the model to be considered monotonic. Recalling equations 44 and 45, the objective function is said to be monotonic if the decision variables for each stage affect the individual stage returns in the same fashion as the way they affect the overall objective function. Since the overall efficiency is an average of the stage efficiencies, the decision variables of each stage affect the individual stage efficiency and the overall efficiency in the same fashion. For example, if the decision variables in the first stage lower the first-stage efficiency, the overall compressor efficiency would also decrease.

Now that separability and monotonicity of the objective function have been established for the two-stage compressor, dynamic programming techniques can be applied to the model. Considerations for the inlet conditions and recurrence relationships can be built for the compressor model. The inlet conditions for both stages were developed in Chapter 2. The only changes in the inlet conditions are the change in rotational velocity of the rotors and the inlet air angles of each stage, namely N, α_1, α_3. These are used to build the recurrence relationship between the stages. Since the rotational velocity is the same for each rotor due to the rotors being attached to the driveshaft, the rotational velocity will be a constant for each stage. The optimal overall efficiency was examined at different rotational speeds to determine the overall optimal operating conditions. As stated, the rotational speed is highly dependent on the requirements of the turbine. Therefore, the compressor's optimal rotational speed may not be accommodated. By using dynamic programming, information can be gained concerning the optimal design for a specific rotational speed in the event that the optimal rotational speed cannot be achieved.

As for the inlet air angles, they are independently chosen by the designer. For this research, the inlet air angles to the first and second stages were constrained to be between -20^o and 20^o. This was done to ensure the flow through the blades would allow the stages to be operational for on-design conditions. With the inlet air angles not dependent on what changes the air undergoes prior to the inlet, the recurrence relation between each stage is only the choice of the air angle.

The approach used for this problem to determine the optimal overall efficiency of the compressor can be expressed as follows. First, discretize the ranges over the upper and lower bounds of the inlet air angles and rotational speed of the rotors. Next, for each inlet condition to stage two, determine the optimal design for stage two:

$$\eta^{2^*} = opt_{\alpha_3,N}(\eta^2 \left[\left(\frac{s}{c}\right)^2_{r,s} ; \alpha_3, N \right] . \tag{49}$$

Then, using each inlet condition to stage one (α_1, N), solve for the optimal design of the first compressor stage that yields the outlet conditions used for inlet conditions to stage two (α_3, N). Calculate the overall objective function for each combination of variables. An overall optimal solution for the dynamic programming method can be determined as:

$$\eta^*(\alpha_1, \alpha_3, N) = \underset{\alpha_1,N}{opt} \left[\frac{1}{2} \left(\eta^1 \left[\left(\frac{s}{c}\right)^1_{r,s} ; \alpha_3, N \right] + \eta^{2^*}(\alpha_3, N) \right) \right] . \tag{50}$$

Examining what stage designs yield the overall optimal efficiency, the design for the first and second stages can then be found. This can be used as the optimal design or trajectory of the first iteration of the dynamic program. A new, narrower range for each variable can be constructed around the trajectory. A similar approach can be taken to determine the new optimal trajectory. Determining the optimal trajectory for each iteration and narrowing the ranges of the variables for each successive iteration allows for the global maximum efficiency to be found.

For each stage, the optimal design had to be determined. Since it was already determined the GRG optimization method was the best method to solve for a single compressor stage, this method was used to determine the optimal single stage designs for given inlet conditions for the dynamic program.

The two-stage model is solved by using the dynamic approach and also by modeling the compressor as a single system (the "composite" model) incorporating the GRG method to solve for the single system to compare against the dynamic programming results. By examining the two methods, the difference can be seen for the two-stage model. For two stages, the benefit of using dynamic programming cannot be observed since the principle of optimality is not applicable for two stages.

It is expected that as more stages are added, the dynamic programming approach will become more efficient than the single system method. There are two reasons to expect this to be true. First, as the number of stages is increased, the size of the single system model increases in complexity since each stage adds five additional variables to the problem, namely spacing for both blade rows, the chord for both blade rows, and the inlet air angle to the stage. In contrast, the dynamic programming approach would add the inlet air angle to the number of variables that could be changed. Also it would add an additional single stage that would need to be optimized. Secondly, as more stages are added, the dynamic programming method will be able to remove from consideration some non-optimal overall compressor designs. Table 4 below shows how the addition of design variables affects the number of additions and comparisons for direct search methods compared to the dynamic programming method. In the table, N is the number of stages in the problem, and J represents the number of decision variables in the problem.

Table 4. Number of additions and comparisons for a direct search method and a dynamic programming method

Direct search

N	J = 2	J = 10	J = 100	J = 1000
2	15	2×10^3	2×10^6	2×10^9
3	47	3×10^4	3×10^8	3×10^{12}
5	319	5×10^6	5×10^{12}	5×10^{18}
10	2×10^4	10^{12}	10^{23}	10^{34}
50	5×10^{16}	5×10^{52}	5×10^{103}	5×10^{154}

Dynamic programming

N	J = 2	J = 10	J = 100	J = 1000
2	9	289	3×10^4	3×10^6
3	15	479	5×10^4	5×10^6
5	27	959	9×10^4	9×10^6
10	66	1809	1.9×10^5	1.9×10^7
50	346	9409	10^6	10^8

Table 4 shows the advantages of using dynamic programming as the number of stages and variables increase, but there is one possible drawback of dynamic programming that needs to be taken into account, the concept dealing with the growth in the number of variables known as the *curse of dimensionality* (Nemhauser,66). This can be described as when the number of variables increases, the total number of possible combinations that need to be examined increases at a rapid rate. Since an additional stage only requires an additional air angle to be considered in the dynamic program, the curse of dimensionality does not pose a threat for a limited number of stages. If the number of stages increases, the number of discretized values used to describe the range of each air angle and rotational speed could be reduced. This would reduce the number possible combinations, but would also require more iterations to obtain a near-optimal solution. When examining multi-stage compressors, this would have to be considered to determine if the reduced set of combinations to be considered would improve the time to solve the problem with the increased number of iterations.

3.4 Conclusion

Two approaches were presented to solve the two-stage compressor model; GRG applied to the composite model, and the dynamic programming method. For a single-stage compressor, the GRG method was considered to be an efficient way to determine the optimal variables for the model. For the two-stage model, the GRG method is still capable of solving the problem, but consideration was given to future work where multiple stages will be examined. In the case of multiple stages, it is likely the GRG method would not be capable of handling all the constraints and solving for all the variables. Dynamic programming was considered to be a viable approach to use in the case of a multi-stage compressor. This research used the dynamic programming method to determine the optimal air angles, rotational speed and spacing-to-chord ratios for each stage. It used the GRG algorithm found in Microsoft Excel solver package to solve for the optimal design of each individual stage. To compare against the dynamic programming method, a composite two-stage compressor model was solved using the GRG method. This was done to determine how close the results from the dynamic programming method tracked the results that the GRG method returned. The results of this analysis are shown in the next chapter.

Chapter 4 - Results

4.1 Introduction

The focus of this research can now be stated as: determining the optimal efficiency of the two-stage compressor, as well as determining how efficient dynamic programming was in solving the problem compared to solving the compressor model using a single composite model. The compressor model was verified through comparisons to examples in Cohen (Cohen, *et al.*, 1985) and Massardo (Massardo and Satta, 1990) as discussed in section 2.6. Once the model was determined to be properly representing the compressor, the dynamic programming method and the single composite model using the GRG method were applied to determine the optimal design of both compressor stages. Results for the optimal design and sensitivity analysis to changes in design variables are shown in the following section. Next, a comparison of both the composite model and the stage-wise model are presented. This is done to determine if dynamic programming method has possible applications for determining the optimal design of a more complex multistage compressor. Finally, there is an examination of the off-design performance of the optimal design as well as other near-optimal designs to determine how the design would perform under off-design conditions.

4.2 On-Design Optimal Results

The model was solved by the dynamic programming method to determine the optimal design of the compressor. The equation to determine the optimal two stage efficiency using dynamic programming is,

$$\eta^*(\alpha_1, \alpha_3, N) = \underset{\alpha_1, N}{opt} \left[\frac{1}{2} \left(\eta^1 \left[\left(\frac{s}{c}\right)^1_{r,s} ; \alpha_3, N \right] + \eta^{2^*}(\alpha_3, N) \right) \right] \tag{51}$$

where $\eta^{2^*}(\alpha_3, N)$ is the optimal efficiency for the second compressor stage for given α_3 and N. The optimal efficiency of stage 2 is calculated from

$$\eta^{2^*}(\alpha_3, N) = \underset{\alpha_3, N}{opt} \left[\eta^2 \left(\frac{s}{c}\right)^2_{r,s} ; \alpha_5, N \right] . \tag{52}$$

As stated in Chapter 3, the stage 2 outlet air angle, α_5, is equal to zero for this research. Discrete values for the inlet decision variables, α_3 and N, were chosen since the variables are continuous. The discrete values were chosen to cover the range of the variables, where the range for each variable corresponds to the bounds placed on the variables as shown in Table 2. This allows the model to solve for the optimal s/c ratios for both the rotor and the stator in stage 2 for each combination of α_3 and N.

Table 5. Stage 2 discrete variable choices for dynamic programming-iteration 1

α_3	-20	-10	0	10	20
N	250	225	200	175	150

Since the outlet air angle was defined to be equal to zero, variations of stage-two outlet conditions are not considered. If the outlet air angle was allowed to vary, the number of possible combinations for stage two would be 125. As presented in developing the compressor model, the outlet conditions of the compressor are the inlet conditions of the combustion chamber. This needs to be considered when designing the compressor. It was seen in this research, and also in Massardo and Satta (Massardo and Satta, 1990), that as the outlet air angle of stage 1 increases, the efficiency of the stage increased. This should hold true for stage 2 as well, although the effect of the change in outlet conditions of stage 2 to the two-stage compressor would not be as large as the effect of changes in the outlet conditions of a single-stage compressor.

The model for the second stage was solved using the GRG method to determine the optimal efficiency for all combinations shown in Table 5. Figure 17 shows how the stage 2 efficiency was affected by changes in α_3 and N.It can be seen that the efficiency of stage 2, shown in Figure 17, is sensitive to the rotational speed of the rotor. The values for the efficiency are saved as the $\eta^{2^*}(\alpha_3, N)$ for each combination of α_3 and N.

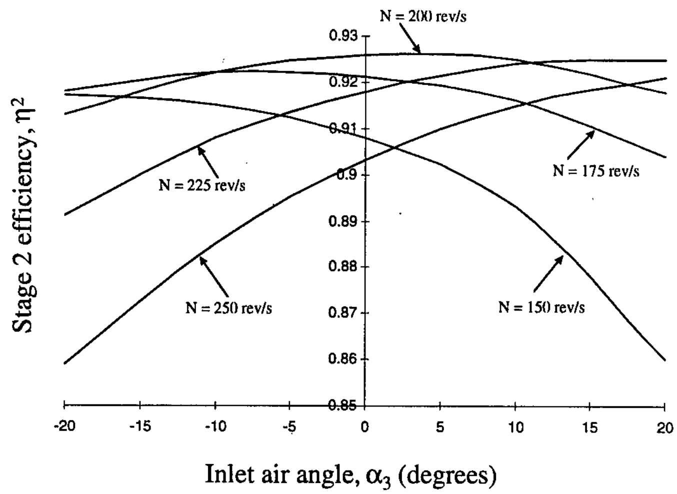

Figure 17. Variations in stage efficiency for stage 2 for inlet variables.

Once the second stage efficiency was calculated for each combination of variables in Table 5, the dynamic program solved for the optimal two-stage compressor efficiency. The dynamic program examined all combinations of inlet variables to stage 1 that yielded the inlet conditions to stage 2 shown in Table 5. Table 6 shows the discrete values of stage 1 inlet conditions and the required outlet conditions of stage 1.

Table 6. Stage 1 discrete variable choices for dynamic programming-iteration 1

inlet					
α_1	-20	-10	0	10	20
N	250	225	200	175	150
outlet					
α_3	-20	-10	0	10	20
N	250	225	200	175	150

An efficiency can be found for stage one using combinations of variables shown in Table 6. Using Equation 51 and the optimal efficiencies of stage 2 for each combination in Table 5, an overall two-stage efficiency is calculated for each combination of variables in Table 6. To determine the optimal efficiency for both stages, Appendix B shows graphs that represent the changes in efficiency for the two-stage compressor. The optimal efficiency is found to be achieved when $\alpha_1 = 10$, $\alpha_3 = -10$, and $N = 200$ rev/s. Figure 18 is taken from Appendix B.1 to show the efficiency at this design.

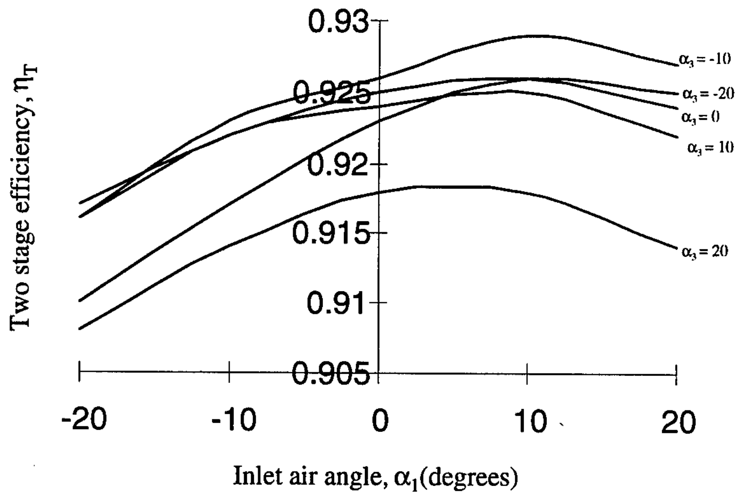

Figure 18. Total compressor efficiency with N = 200 rev/s.

During this first iteration of the dynamic programming method, information was collected about the optimal spacing-to-chord ratios, s/c, for each blade row for all the combinations of variables in Table 6. They were used to create the graphs shown in Appendix B.2 that show how the s/c ratios of the stage 1 rotor compared to the stage 1 stator. The graph showing the optimal spacing-to-chord ratios for the optimal air angles and rotational speed presented earlier is shown in Figure 19. The graph shows the changes in the rotor and stator s/c ratios as α_1 varies from its lower bound to its upper bound while α_3 and N are fixed to their optimal values.

As α_1 increases, the s/c ratios for the rotor and the stator become approximately the same.

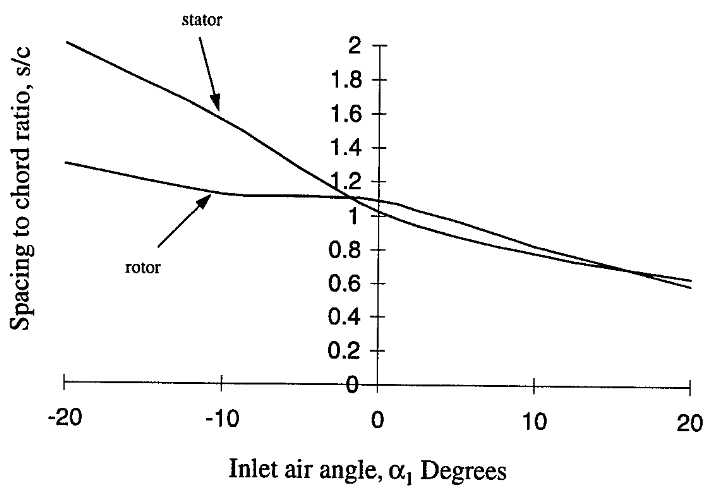

Figure 19. Spacing to chord ratios for stage 1 blade rows for optimal design.

For the optimal design of $\alpha_1 = 10$, $\alpha_3 = -10$, and $N = 200$ rev/s, the stage 1 rotor was $(s/c)^1_r = 0.827$ and the stator was $(s/c)^1_s = 0.778$. This yielded a two-stage efficiency of $\eta^*_T = 0.9286$.

The optimal spacing-to-chord ratios for the second stage can be found by examining the $\eta^{2^*}(\alpha_3, N)$ for $\alpha_3 = -10$ and $N = 200$ rev/s. They are found to be $(s/c)^2_r = 1.022$ and $(s/c)^2_s = 0.881$. This design is taken to be the optimal trajectory for the first iteration of the dynamic program. The second iteration decreases the range of the variables and centers the range on the value of the first iteration trajectory. For the second stage, the discrete values representing the design variables ranges are seen in Table 7.

Table 7. Stage 2 descrete variable choices for dynamic programming-iteration 2

α_3	-15	-10	-5
N	205	190	175

Only three discrete values were chosen to decrease the number of calculations to determine the optimal design for each stage. The same approach is taken as with the first iteration; the efficiency for each combination of variables is calculated and saved as $\eta^{2^*}(\alpha_3, N)$. Then the efficiency for stage 1 is determined for each combination of the variables shown in Table 8.

Table 8. Stage 1 discrete variable choices for dynamic programming-iteration 2

inlet				**outlet**			
α_1	5	10	15	α_3	-15	-10	-5
N	205	190	170	N	205	190	175

Using the GRG method for each stage independently, the optimal efficiencies for the compressor were found for each combination of variables in Table 8. The overall optimal efficiency was found to be $\eta_T^* = 0.9287$ and the optimal design was found to be $\alpha_1 = 10$, $\alpha_3 = -10$, and $N = 209.6$ rev/s. This yielded the spacing-to-chord ratios for each blade row given in Table 9.

Table 9. Dynamic Programming optimal spacing-to-chord ratios

$$\begin{array}{llll} (s/c)_r^1 = & 0.884 & (s/c)_r^2 = & 1.071 \\ (s/c)_s^1 = & 0.778 & (s/c)_s^2 = & 0.908 \end{array}$$

The dynamic program was stopped here since the efficiencies of the two successive iterations were less than the tolerance of 0.001. The efficiency in the second iteration is taken to be the optimal two-stage compressor design.

To determine how efficient the dynamic program was at determining total efficiency, a composite model of both stages was developed to solve for the optimal design using the GRG method on the two-stage compressor. Eight separate runs were done, starting from different initial decision variables, to find the global optimal efficiency in the bounded design space. The first six runs converged to local optimal efficiencies at lower values than efficiency found by the dynamic program. The seventh and eighth runs converged to approximately the same design as the dynamic program returned as the optimal answer. The optimal design found by the GRG method is given in Table 10.

Table 10. Optimal design variables determined from GRG method

$\alpha_1 =$	13.2	$(s/c)_r^1 =$	.842
$\alpha_3 =$	−7.07	$(s/c)_s^1 =$	.705
$N =$	209.6	$(s/c)_r^2 =$	1.18
		$(s/c)_s^2 =$	0.858
	$\eta_T^* =$	0.9293	

After two iterations the stage-wise model yielded an overall compressor efficiency of 0.9287; therefore, the percent error of the dynamic programing method (after two iterations) from the optimal design determined by using single composite model is less than 1%. This shows the optimal design found by the dynamic program can be considered an optimal design. The blade angles for the optimal design in the dynamic program are shown in Figure 20.

4.3 Dynamic Programming Compared to Composite Single System Model

Both models converged to approximately the same design. One question raised is: which method was more efficient in determining the optimal design? The benefits for using dynamic programming were not seen in solving the two-stage compressor since the principle of optimality trivially applies to only two stages. For two stages, the GRG method was able to solve for the optimal design since there were only 5 decision variables and approximately 100 constraints, thus it was the more efficient method to use to solve for the optimal design. The GRG method used 8 iterations where a total of 263 iterations of the GRG algorithm were performed. In contrast, the dynamic programming method used 186 iterations where a total of 968 iterations of the GRG algorithm were performed to determine the optimal design. The number of GRG method iterations show it is more efficient to use the composite single system model rather than the stage-wise model to determine the optimal design for the two-stage compressor.

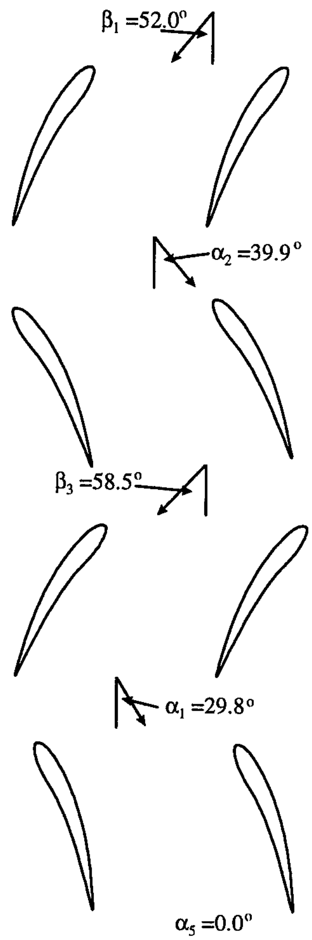

Figure 20. Blade angles for optimal design.

The second focus of this research was to determine which optimization method would be more efficient to use to determine the optimal design of a multistage compressor. If a composite model was developed for the entire multistage compressor, the GRG method would have to include a decision variable for the rotational velocity plus 2 decision variables for each stage of the compressor. Also, the model would have to include approximately 50 constraints for each stage of the compressor. After just a few stages, the number of constraints and decision variables would make the problem impossible to solve using the GRG method. Using the dynamic programming method, each stage of the compressor adds an additional decision variable representing the inlet angle to the stage. In addition, each stage would have to be individually solved for its optimal design based on certain inlet and outlet conditions for each stage. For example, a four stage compressor would require the GRG method to solve for 13 decision variables with over 200 constraints. This would be a very difficult problem to solve by nonlinear programming methods.

In contrast, dynamic programming would have 5 decision variables and would need to use the GRG method to solve 4 individual stages for specific conditions. If the decision variables were represented by three discrete values for an iteration, as was the case for the second iteration in the two-stage model, 360 combinations of variables would have to be considered for each iteration. Multiple iterations could be run to determine the optimal efficiency of the compressor.

Figure 21 shows how the number of iterations of the GRG optimization method need to be performed as the number of stages increase. The line segment representing the number of required iterations that increases most rapidly corresponds to using a stage-wise optimization approach and choosing 5 discrete values for each inlet variable (as was done for the first iteration of the two-stage compressor model using the dynamic programming approach). The middle line represents the number of GRG iterations it would take to determine an optimal design if the compressor was modeled as a single composite system. This line is based on the increase in the number of iterations the GRG method needed to solve the two-stage compressor compared to solving for the design of a single-stage compressor. The curve that increases at the slowest rate represents the number of GRG iterations required to achieve an optimal design if 3 discrete values of the inlet decision variables were chosen for each iteration (as was done in the second iteration of the two-stage compressor). This result shows that as the number of stages increases, a stage-wise approach to determine an optimal design becomes more efficient than maintaining the model as a single composite system.

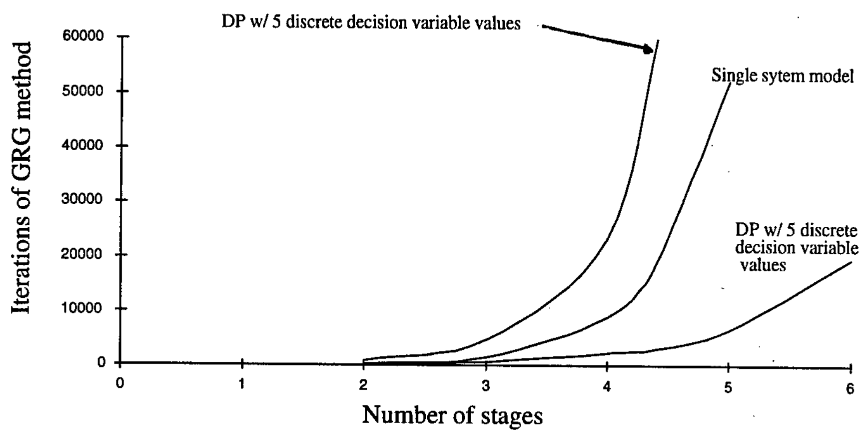

Figure 21. Comparision of iterations of GRG method required to obtain an optimal design as the number of stages increase.

As the number of stages increases, the size and complexity of the single composite system model would become large and overwhelm the GRG method. In contrast, the dynamic programming method would be capable of including an additional decision variable for each stage inlet air angle and an additional single stage optimization model to determine the total efficiency of the compressor.

4.4 Off-design of Two-Stage Compressor

The optimal design was found for the on-design conditions of the compressor, but a major question asked when designing a compressor is: how well would the compressor operate under off-design conditions? Since the compressor may be operating at off-design for the majority of the time, this needs to be considered in the compressor design.

It would be time consuming and not practical to complete an off-design analysis for all the designs considered in the dynamic program, but it is beneficial to examine some of the vital designs to determine how changes in the compressor design affect the off-design performance. Probably the most important design to consider is the optimal design found by the dynamic program; it is important to know how it performs under off-design conditions. For the optimal design, the first stage had a higher efficiency than the second stage. Two other designs were chosen for off-design analysis based on the design characteristics of their stage efficiencies. One design had an equal efficiency for each stage, and the other had a higher efficiency for the second stage than for the first stage. An off-design analysis was completed for these three cases according to the off-design equations in Appendix C. The results are shown in Figure 22.

The optimal design, case 1 in the graph, continues to be the optimal design when the inlet air angle is less than the inlet blade angle. As the inlet air angle increases to be larger than the inlet blade angle, the design with equal efficiencies in both stages becomes the better design. Since case 1 and case 3 have similar characteristics (both cases have a stage with a higher efficiency than the other), and the off-design conditions lower the efficiency for these cases more than in case 2 for positive differences in incidence angle, it is concluded the difference in stage efficiencies plays an important role in the off-design performance. For a negative difference in incidence angle, the original optimal design remains optimal. The designer can use this information along with the amount of time the compressor would be subjected to certain air angles, to determine which case would be the better design choice.

The cases in Figure 22 represent each design considered for off-design analysis in the following way:

case 1 = The optimal design found in section 4.2.

case 2 = The design with equal efficiency for each stage.

case 3 = The design with higher stage 2 efficiency.

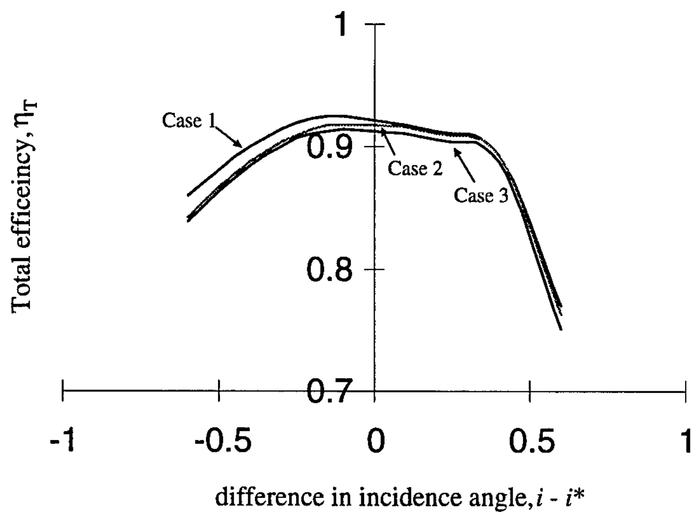

Figure 22. Off-design analysis for specified design choices.

Obviously, all results presented are based on the operating conditions and limitations of the model. These results, however, do give an initial starting point for the designer to calculate on-design performance for a multistage compressor. The model can be used to determine the optimal spacing-to-chord (s/c) ratios and the optimal Mach number. The degree of reaction, as well as the efficiency for each stage of the optimal design can also be found.

Chapter 5 - Summary, Conclusions, Recommendations

5.1 Summary

This research was focused on developing a pitchline model for a two-stage compressor, and using an optimization method to solve for the optimal on-design blade geometry for each blade row. The compressor was modeled and optimization methods were analyzed to determine which one(s) would be suited for use in solving for the optimal compressor design. To accomplish this, a model representing the two stage compressor was formulated with an objective of maximizing the overall efficiency of the two-stage compressor. Bounds were placed on the rotational speed of the rotor and the blade geometry to ensure the compressor would maintain operational capabilities for the optimal design.

To solve for the optimal compressor design, only nonlinear optimization methods were considered since the compressor model contained nonlinear constraints and objective function. Also, since the compressor is able to be separated into individual stages, a dynamic programming method was also considered.

The GRG method was considered to be an efficient method for solving single and two-stage compressor models, but it would likely become impractical to use for multistage compressors due to the growth in the number of design variables and constraints. For multiple stages the incorporation of dynamic programming methods was considered necessary to solve for the optimal design. The dynamic programming method simplified the overall model by separating each stage and linking them using the outlet conditions from the previous stage as the inlet conditions to the subsequent stage. Each stage could then be solved for its optimal design using the GRG method, providing a solution for the optimal design of a multistage compressor

This research solved the two-stage compressor model using both the dynamic programming method, where each stage was solved by the GRG method, and the GRG method applied to the composite model, to determine the efficiency of each method in obtaining the optimal design. Also, information was obtained concerning the optimal inlet blade angles for each stage and the optimal s/c ratios for each blade row. Both methods converged to approximately the same design, with the difference in the efficiency being less than 1%. The optimal rotational speed of the rotor was determined to be 209.6 m/s. Analysis was performed to determine what effect deviations would have on the compressor efficiency if the rotational speed differed from the optimal speed. For the optimal design, the s/c ratio for each blade row is seen in Figure 11 and yielding the blade angles shown in Figure 20.

Table 11. Dynamic Programming optimal spacing-to-chord ratios

$(s/c)_r^1 =$	0.884	$(s/c)_r^2 =$	1.071
$(s/c)_s^1 =$	0.778	$(s/c)_s^2 =$	0.908

For the two-stage compressor, the single composite system approach is able to solve the optimal model design more efficiently than the dynamic programming method, but two stages is not enough to take advantage of the principle of optimality as exploited by dynamic programming methods. As the number of stages increase, the number of design variables added to the compressor model would make solving the model by the composite system method alone impractical. Therefore, as the number of stages increase, dynamic programming becomes more attractive. In the dynamic program, the GRG method can be used to solve for the optimal design variables for each stage.

5.2 Conclusions and Recommendations

This thesis presented the necessary assumptions and conditions to develop a mathematical model of a multi-stage compressor. A model was developed for a two stage compressor for use in determining optimization methods that would be useful in determining optimal designs for larger compressor models. Results were used to determine whether it was better to model the compressor as a single composite model or to use stage-wise decomposition of the compressor. The two-stage results showed solving the compressor model as a composite system was more efficient than separating the compressor into its stages and using a dynamic programming approach.

Since compressors used in aircraft are composed of more than two stages, comparisons were made between the stage-wise optimization methods and solving the optimal design of the compressor as a single composite system. It was concluded from this research that as the number of stages increases in the compressor, decomposing the compressor into its individual stages and incorporating a dynamic programming method would be a more feasible approach to determine an optimal compressor design. This was based on the increase in model complexity for both approaches as the number of stages increased.

This conclusion is based on the on-design results of the two-stage compressor; no attempt was made to include the compressor's off-design performance into the optimization of the compressor. Since the compressor may operate at off-design a great portion of the time, this is an important consideration that needs to be taken into account in the compressor design. It would be valuable to include the off-design performance some way as a part of the optimization of the compressor, to improve the performance of the compressor over all ranges of off-design (trying to incorporate the off-design performance is suggested as future research in the next section).

It is recommended to incorporate a dynamic programming approach to the compressor as the number of stages increases in order to simplify the mathematical model. By doing this, an optimal compressor design might be determined for a multiple stage compressor. Also, it would be beneficial to incorporate the off-design performance of the compressor into the model in some fashion to determine the optimal design overall operating conditions, and not only on-design conditions

5.3 Future Research

Since this is one of the first attempts to incorporate optimization methods to solve for an optimal compressor design, there are many possibilities for future research. They can be separated into two areas: model improvement, and addition of stages.

In the area of model improvement, there are many things to consider for future analysis. The model in this research was limited by the axial velocity of the air being constant. While this gives an initial estimate of the flow, in reality the axial velocity varies across the blade rows from stage to stage. By keeping the axial-flow velocity constant, the size of the annulus flow area is forced to be a specific size due to the conservation of momentum of the air flow. This could force the compressor, and hence the engine's weight, to become detrimental to aircraft performance. Also, it would be beneficial to use the objective function from Massardo and Satta (Massardo and Satta, 1990) to examine the multistage compressor. Their objective function used a multivariable approach to include the area of flow and the off-design performance of the compressor. By doing this, the problem of the compressor being too large can be accounted for by the area of flow as modeled in the objective function. The off-design performance can be used to examine how other blade geometries affect the performance of the compressor. Referring to Appendix C, the analysis of the off-design performance included the camber angle and the blade angles.

For the possibility of additional stages being added to the compressor, it would be useful to determine at what point the GRG method no longer becomes an option to solve for the optimal compressor design. Some work (Beknev, *et al.*, 1991) is being performed to find new methods that might be more powerful than the GRG method in solving the compressor design problem. More analysis should include the use of a dynamic programming method to solve for the optimal (or near-optimal) design of a multistage compressor.

As stated, this is one of the first attempts found in the open literature that approaches the problem of modeling a compressor in order to determine the optimal design. More work needs to be done in the areas of compressor modeling to obtain a model that is truly representative of a multistage compressor. This research did show the ability of using optimization techniques to improve a compressor design to achieve a higher efficiency.

APPENDIX A - Approximation of Objective Function

This appendix shows how the approximation used in the dynamic programming method compares to the actual objective function. The approximation used in the dynamic programming method to determine the optimal design is given by

$$\eta_T = \frac{1}{2}(\eta^1 + \eta^2) \tag{53}$$

The table shows the percent error between the actual efficiency and the approximation given in Equation 53. The actual efficiency is found by

$$\eta_T = 1 - \left(\frac{\omega_T^1 + \omega_T^2}{\Delta P_{th}^1 + \Delta P_{th}^2}\right) \tag{54}$$

Stage 1 losses, ω_T^1	ΔP_{th}^1	Stage 2 losses, ω_T^2	ΔP_{th}^2	actual η_T	approximate η_T	% error
.035	.35	.035	.35	0.9	0.9	0
.035	.35	.035	.55	0.9222	0.91818	0.4381
.035	.35	.055	.35	0.87143	0.87143	0
.035	.35	.055	.55	0.9	.09	0
.035	.55	.035	.35	0.9222	0.91818	0.4381
.035	.55	.035	.55	0.93636	0.93636	0
.035	.55	.055	.35	0.9	0.8896	1.154
.035	.55	.055	.55	.91818	.91818	0
.055	.35	.035	.35	.87129	.87129	0
.055	.35	.035	.55	.9	.8896	1.154
.055	.35	.055	.35	.84286	.84286	0
.055	.35	.055	.55	.87778	.87143	0.723
.055	.55	.035	.35	0.9	0.9	0
.055	.55	.035	.55	0.91818	0.91818	0
.055	.55	.055	.35	0.87778	0.87143	0.723
.055	.55	.055	.55	.9	.9	0

The data shows the approximation for the efficiency is off at most 1.154% from the value of the actual efficiency. This suggests the approximation would be a valid choice to use for the dynamic programming method to solve the problem.

APPENDIX B - On-Design Optimal Graphs

B.1 Efficiency Variation for Changes in Rotational Speed

While performing the first iteration of the dynamic program algorithm, information was gathered concerning the efficiency as the inlet air angles were specified as discrete values. The graphs below show the way the inlet air angles, $\alpha 1$ and $\alpha 3$, change the efficiency. For each graph, the rotational velocity is held to one of the discrete values used in the first iteration of the dynamic programming algorithm. This information could be helpful in determining the optimal rotational velocity to try to achieve for given inlet air angles. The graph shown here has the rotational velocity fixed at 225 rev/s. The irregularity of the curves for $\alpha_3 = 20$ and $\alpha_3 = -20$ could be due to

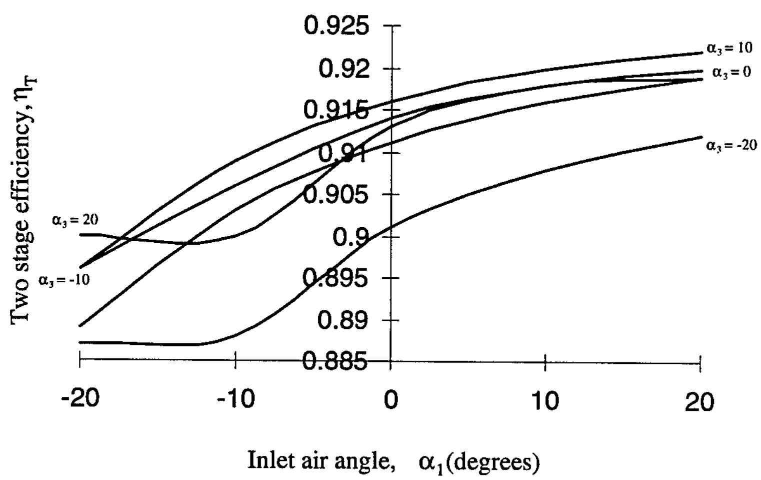

Figure 23. Efficiency variations for N = 225 rev/s.

those being the extreme values of α_3. It is seen from the figure above that the efficiency increases as the inlet air angle, α_1, increases. This can be accounted for by the decrease in shock losses for increased α_1. The relative air angle, β_1, decreases resulting in a lower relative Mach number. The lower inlet Mach number lowers the shock losses through the rotor. Figure 24 represents the efficiency changes for a rotational speed of 200 rev/s This Figure gives the overall optimal air angles.

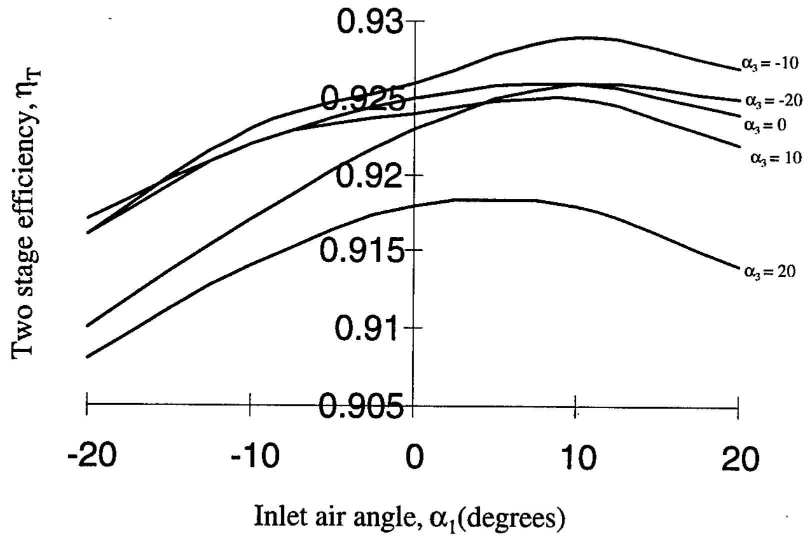

Figure 24. Efficiency variations for N = 200 rev/s.

They are seen at $\alpha_1 = 10$ and $\alpha_3 = -10$. Also seen from Figure 24, the efficiency doesn't keep increasing as α_1 increases. Figure 25 and Figure26 show the same general configuration as Figure 24, only the efficiencies are not at the same value as for the efficiency with a rotational velocity of 200 rev/s. This is why 200 rev/s was taken as the optimal rotational velocity for the first iteration.

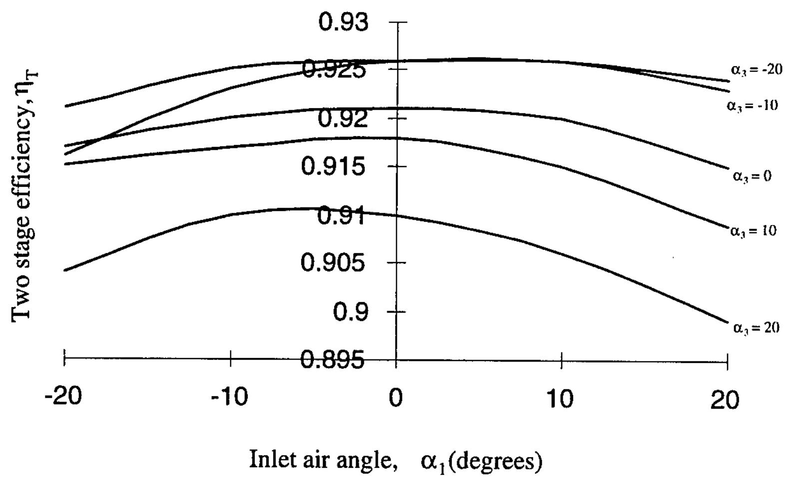

Figure 25. Efficiency variations for N = 175 rev/s.

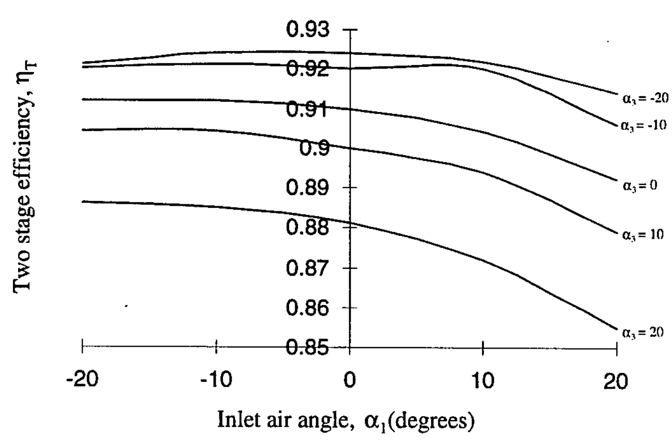

Figure 26. Efficiency variations for N = 150 rev/s.

B.2 Spacing-to-Chord Ratio Graphs

While performing the first iteration of the dynamic programming algorithm, another collection of information is used to aid in design for conditions other than the problem modeled. Graphs can be created for the optimal spacing-to-chord ratios (s/c) for specific inlet air angles and rotational velocities. Since the optimal rotational speed was determined from Appendix B.1 to be 200 rev/s, the optimal s/c ratios will be graphed for varying inlet air angles α_1 for a rotational speed of 200 rev/s.

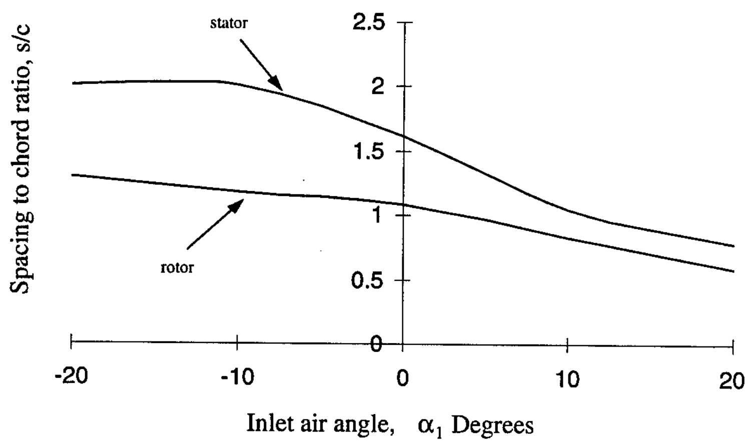

Figure 27. Spacing-to-chord ratios for air angle of -20 degrees.

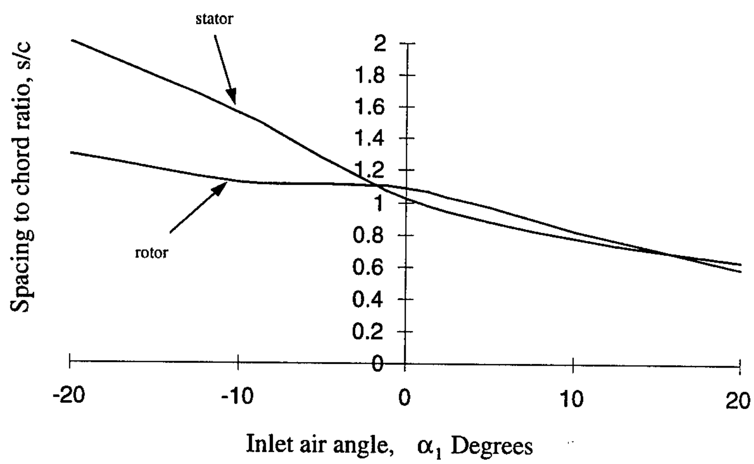

Figure 28. Spacing-to-chord ratios for air angle of -10 degrees.

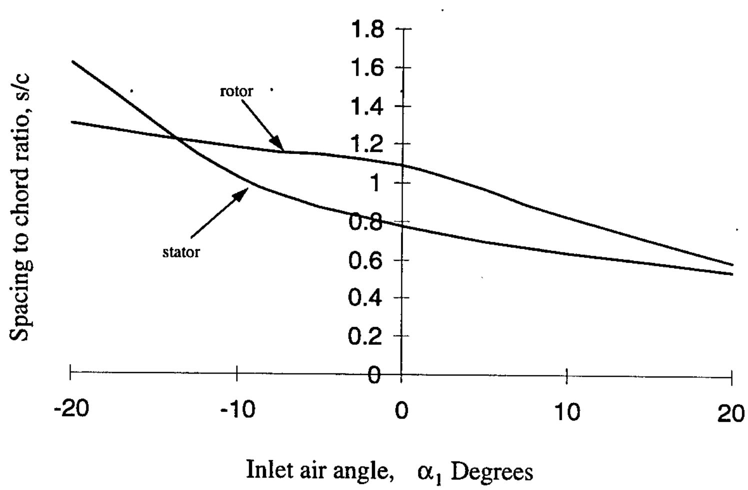

Figure 29. Spacing-to-chord ratios for air angle of 0 degrees.

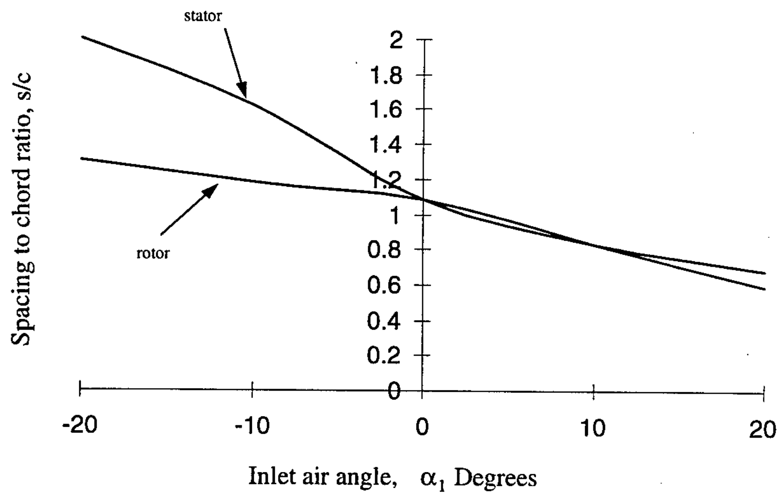

Figure 30. Spacing-to-chord ratios for air angle of 10 degrees.

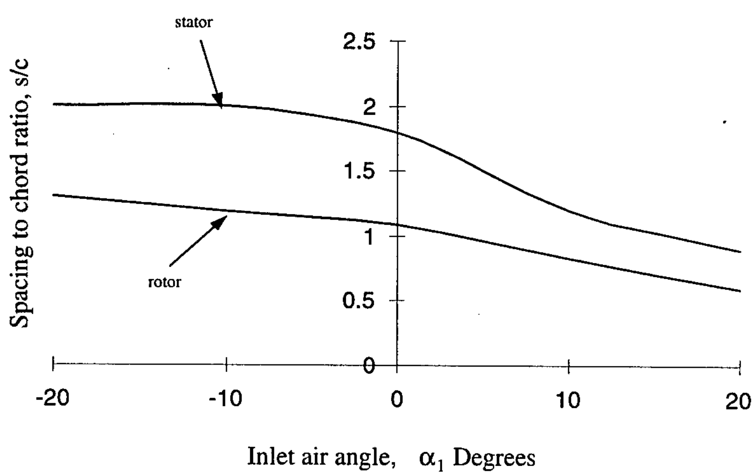

Figure 31. Spacing-to-chord ratios for air angle of 20 degrees.

APPENDIX C - Off-Design Formulation

In order to determine the off-design performance, calculations of the losses and pressure rise over each blade row had to be considered for a variety of inlet incidence angles into stage 1. This was taken from Horlock (Horlock, 1973) as an initial calculation of off-design performance. In Horlock, only a single stage is analyzed and an assumption of 50% degree of reaction is used to determine the off-design performance of the single stage compressor. Below is the formulation used to determine the off-design performance of the two stage compressor considered in this research. First, the model was solved for the optimal design for each combination of operating conditions shown in Chapter 5. Once the optimal s/c ratios are determined for each combination, analysis of off-design was performed for the three cases shown in Chapter 5.

Starting at the inlet to stage 1, the blade angles were used to determine the deflection of the blade across the rotor by

$$\epsilon_r^1 = \beta_1 - \beta_2. \tag{55}$$

This value is then used along with the spacing to chord (s/c) ratio for the blade design to calculate the nominal deviation, δ_r^*, and β_2. The formula for the nominal deviation is

$$\delta_r^{1^*} = m\epsilon_r^1\sqrt{\frac{s}{c}} \tag{56}$$

where $m = 0.23\left(\frac{2a}{c}\right)^2 + 0.1\left(\frac{\beta_2^*}{50}\right)$. the value of a represents the distance from the leading edge of the blade to the point of maximum camber along the blade. Since that is blade specific, and the type of blade for this research was not defined, an assumption that $m = 0.19$ was is (Cohen, *et. al.*, 85).

Once a value of $\delta_r^{1^*}$ is found, the outlet air angle is found by

$$\beta_2^* = \beta_2 + \delta_r^{1^*}. \tag{57}$$

Figure 32 is used with β_2^* to determine the nominal deflection, ϵ^*, for the rotor Now the inlet air

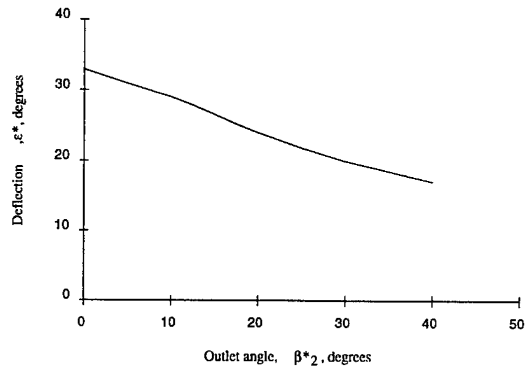

Figure 32. Deflection angle of blade based on nomimal outlet angle.

angle can be determined by,

$$\beta_1^* = \beta_2^* + \epsilon^*. \tag{58}$$

Using the inlet air angle, β_1^*, and the inlet blade angle,β_1, the nominal incidence (i^*)is calculated as $\beta_1^* - \beta_1$.

The air angles for the first stage rotor have been determined; now the stator air angles can be found. Since the stator inlet angle is the same as the outlet angle of the rotor, the stator inlet blade angle is

$$\alpha_2^* = \alpha_2 + \delta_r^{1^*}. \tag{59}$$

The outlet air angle for the stator is found by subtracting off the deflection of the stator from the stator inlet angle, α_2^*. The deflection for the stator is found the same way as the deflection for the rotor, only the blade deflection is found by

$$\epsilon_s^1 = \alpha_2 - \alpha_3. \tag{60}$$

For successive stages following the first stage, the same method was taken. Since the stator was stationary, the deviation was considered negligible. That allowed the air angles for the rotor of the following stage to be calculated in the same fashion as the first stage.

The efficiency of each design analyzed for the two stage compressor was then calculated as the incidence angle varied using

$$\frac{i - i^*}{\epsilon} = x \tag{61}$$

where x was changed between -0.6 and 0.6 in this research to account for different off-design conditions.

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Vita

Second Lieutenant Shawn A. Miller was born on 20 May 1972 in Flagstaff, Arizona. He graduated from Wickenburg High School in 1990. In 1996, he accepted a reserve commission in the Air Force after completing the ROTC program and earning a Bachelor's of Science degree in Mechanical Engineering from Northern Arizona University in Flagstaff, AZ. Lieutenant Miller was a direct accession to the Graduate School of Engineering at the Air Force Institute of Technology in June 1996. He is married to the former Melissa Vance of Davis California and has one daughter, Kira Lorene.

Shawn A. Miller

9719 Copperway

San Antonio TX 78109

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Air Force Institute of Technology
2750 P. Street
WPAFB OH 45433

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AFRL/POTF
Bldg 18
1950 5th Street
WPAFB OH 45433

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co-advisor: Dr. James Chrissis ph # (937)255-6565x4338 email: jchrissi@afit.af.mil
co-advisor: Dr. Paul King ph # (937)255-3636x4628 email: pking@afit.af.mil

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13. ABSTRACT *(Maximum 200 words)*

The development of jet engines has become an integral part of maintaining air superiority. In order to achieve the most advanced engine, research has turned to traditional optimization methods to aid in creating new engine designs. To develop simplified mathematical models representative of the engine, the engine can be separated into its components. A jet engine has three major elements, the compressor, combustion chamber and turbine. This research attempts to make an initial analysis of a two stage compressor to determine values of blade angles and spacing to chord ratios for both stages that produce the highest possible efficiency for the overall two stage compressor. A pitch-line model is developed for a two stage compressor and is used in conjunction with a optimization method to solve for the on-design air angles and spacing to chord ratios. The results of the model were compared to examples available in current literature to ensure the model properly represents a compressor stage. The off-design performance of the results was calculated to determine how the designs for on-design operated under off-design conditions. Since practical compressors are made up of many stages, consideration is given to which optimization method would be most useful in solving for a multistage compressor.

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